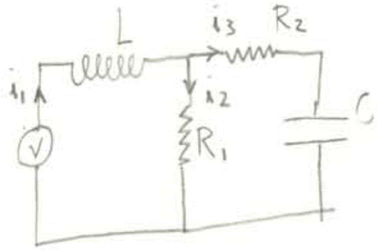
EDI - Prova 3

2010/1

1. Considere o circuito abaixo, com $R_1 = 10\Omega$, $R_2 = 5\Omega$, $L = 1H$, $C = 0.2F$ e

$$v(t) = \begin{cases} 120, & 0 \le t < 2, \\ 0 & 2 \le t \end{cases} \qquad (1.1)$$

Suponha que $i_2(0) = 0$, $i_3(0) = 0$. Determine $i_1(t)$.



$i_2 + i_3 = i_1$

Solução As equações para as duas malhas são dadas por.

$$L\frac{di_1}{dt} + R_1 i_2 = 120\left(1 - H_2(t)\right) \qquad (1.2)$$

$$R_2(i_1 - i_2) + \frac{1}{C}\int_0^t (i_1 - i_2)\,d\tau - R_1 i_2 = 0 \qquad (1.3)$$

Fazendo a transformada de Laplace destas equações obtemos

$$\begin{cases} LsI_1 + R_1 I_2 = 120\left(\frac{1}{s} - \frac{e^{-2s}}{s}\right) & (1.4) \\ R_2(I_1 - I_2) + \frac{1}{Cs}(I_1 - I_2) - R_1 I_2 = 0 & (1.5) \end{cases}$$

$$-I_2\left(R_2 + \frac{1}{Cs}\right) - R_1 I_2 = -I_1\left(R_2 + \frac{1}{Cs}\right) \qquad (1.6)$$

$$I_2\left(R_1 + R_2 + \frac{1}{Cs}\right) = I_1\left(R_2 + \frac{1}{Cs}\right) \qquad (1.7)$$

$$LsI_1 + R_1\frac{\left(R_2 + \frac{1}{Cs}\right)}{\left(R_1 + R_2 + \frac{1}{Cs}\right)} I_1 = \frac{120}{s}\left(1 - e^{-2s}\right) \qquad (1.8)$$

$$I_1\left[Ls + R_1\frac{R_2 + \frac{1}{Cs}}{R_1 + R_2 + \frac{1}{Cs}}\right] = \frac{120}{s}\left(1 - e^{-2s}\right) \qquad (1.9)$$

$$I_1\left(s+10\,\frac{5+\frac{1}{0.2s}}{15+\frac{1}{0.2s}}\right)=\frac{120}{s}\left(1-e^{-2s}\right)$$

$$\mathcal{L}^{-1}\left[e^{-cs}F(s)\right]=u_c(t)\,f(t-c)$$

$$I_1\left(s+10\,\frac{s+1}{3s+1}\right)=\frac{120}{s}\left(1-e^{-2s}\right)$$

$$I_1\left(\frac{3s^2+s+10s+10}{3s+1}\right)=\frac{120}{s}\left(1-e^{-2s}\right)$$

$$I_1=\frac{(3s+1)\left(1-e^{-2s}\right)120}{\left(3s^2+11s+10\right)s}=\frac{(3s+1)\left(1-e^{-2s}\right)120}{s(s+2)(3s+5)}$$

$$=120\left(1-e^{-2s}\right)\left[\frac{12}{5(s+5/3)}-\frac{5}{2(s+2)}+\frac{1}{10s}\right]$$

$$=\left[288\,\frac{1}{s+5/3}-\frac{300}{s+2}+\frac{12}{s}\right]\left(1-e^{-2s}\right) \qquad (1.10)$$

$$i_1=288\,\mathcal{L}^{-1}\left[\frac{1}{s+\frac{5}{3}}\right]-300\,\mathcal{L}^{-1}\left[\frac{1}{s+2}\right]+12\,\mathcal{L}^{-1}\left[\frac{1}{s}\right]$$

$$+288\,\mathcal{L}^{-1}\left[e^{-2s}\frac{1}{s+\frac{5}{3}}\right]-300\,\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s+2}\right]+12\,\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s}\right] \qquad (1.11)$$

$$i_1(t)=288\,e^{-\frac{5}{3}t}-300e^{-2t}+12+288\,u_2(t)\,e^{-\frac{5}{3}(t-2)}-300\,u_2(t)\,e^{-2(t-2)}$$

$$+12\,u_2(t) \qquad (1.12)$$

$$=12\left(1+u_2(t)\right)+u_2(t)\left[-300e^{4}\,e^{-2t}+288\,e^{\frac{10}{3}}\,e^{-\frac{5}{3}t}\right]+288\,e^{-\frac{5}{3}t}$$

$$-300e^{-2t}, \qquad (1.13)$$

2. Determine a solução em série de potências do PVI

$$\left(x^2+3\right)y''+xy'+2y=0,\quad y(0)=1,\quad y'(0)=-2 \qquad (2.1)$$

Determine ao menos 6 termos não nulos e estime o raio de convergência da solução.

Solução $$y = \sum_{n=0}^{\infty} c_n x^n \qquad (2.2)$$

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1} \quad , \quad y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} \qquad (2.3)$$

$$(x^2+3)\sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} + x\sum_{n=1}^{\infty} c_n n x^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n = 0 \qquad (2.4)$$

$$\sum_{n=2}^{\infty} c_n n(n-1) x^n + 3\sum_{n=2}^{\infty} c_n n(n-1) x^{\overbrace{n-2}^{k}} + \sum_{n=1}^{\infty} c_n n x^n + 2\sum_{n=0}^{\infty} c_n x^n = 0 \qquad (2.5)$$

$$\sum_{k=2}^{\infty} c_k k(k-1) x^k + 3\sum_{k=0} c_{k+2}(k+2)(k+1) x^k + \sum_{k=1} c_k k x^k + 2\sum_{k=0}^{\infty} c_k x^k = 0$$

$$\sum_{k=2}^{\infty}\Big[k(k-1)c_k + 3c_{k+2}(k+2)(k+1) + c_k k + 2c_k\Big]x^k \qquad (2.6)$$

$$+ 2\cdot 3 c_2 + 2c_0 \qquad + (3\cdot 3\cdot 2\cdot c_3 + c_1 + 2c_1)\,x = 0 \qquad (2.7)$$

Portanto,

$$c_2 = -\frac{c_0}{3} = -\frac{1}{3} \qquad (2.8)$$

$$c_3 = \frac{1}{18}(-3c_1) = -\frac{c_1}{6} = -\frac{(-2)}{6} = \frac{1}{3} \qquad (2.9)$$

$$3c_{k+2}(k+2)(k+1) = c_k\left[k(k-1) + k + 2\right] = c_k\left[k^2+2\right]$$

$$c_{k+2} = -\frac{k^2+2}{3(k+2)(k+1)}c_k \quad , \quad k = 2, 3, \ldots \qquad (2.10)$$

$k=2$, $$c_4 = -\frac{6}{3\cdot 4\cdot 3}c_2 = -\frac{6}{3\cdot 4\cdot 3}\left(-\frac{1}{3}\right) = +\frac{1}{18} \qquad (2.11)$$

$k=3$, $$c_5 = -\frac{9+2}{3(5)(4)}c_3 = -\frac{11}{180} \qquad (2.12)$$

Portanto, com raio de convergência $\rho = \sqrt{3}$, a solução

$$y(x) = 1 - 2x - \frac{1}{3}x^2 + \frac{1}{3}x^3 + \frac{1}{18}x^4 - \frac{11}{180}x^5 + O(x^6). \qquad (2.13)$$

3. Utilize séries de potências para resolver o PVI

$$\left.\begin{array}{l} y'' + \cos x\, y' + xy = 1, \\ y(0) = 1,\ y'(0) = -1 \end{array}\right\} \qquad (2.14)$$

Solução Supomos

$$\phi(x) = y(x) = \sum_{n=0}^{\infty} c_n x^n \qquad (2.15)$$

$$\phi''(x) = 1 - \cos x\, \phi' - x\phi \qquad (2.16)$$

$$2a_2 = \phi''(0) = 1 - \phi'(0) = 2, \qquad a_2 = 1$$

$$\phi'''(x) = \sin x\, \phi' - \phi - x\phi' - \cos x\, \phi'' \qquad (2.17)$$

$$3!\,a_3 = \phi'''(0) = \phi(0) + \phi''(0) = -1-2 = -3, \qquad a_3 = -\frac{1}{2}$$

$$\phi^{IV}(x) = \cos x\, \phi' + \sin x\, \phi'' - \phi' - \phi' - x\phi'' + \sin x\, \phi'' - \cos x\, \phi''' \qquad (2.18)$$

$$4!\,a_4 = \phi^{IV}(0) = \phi'(0) - 2\phi'(0) - \phi'''(0) = -\phi'(0) - \phi'''(0)$$

$$= 1 - (-3) = 4$$

$$a_4 = \frac{4}{4\cdot 3\cdot 2} = \frac{1}{6}$$

$$\phi^{V}(x) = -\sin x\, \phi' + \cos x\, \phi''$$
$$+ \cos x\, \phi'' + \sin x\, \phi''' - 2\phi'' - \phi'' - x\phi''' + \cos x\, \phi'' + \sin x\, \phi''' + \sin x\, \phi'''$$
$$- \cos x\, \phi^{IV} \qquad (2.19)$$

$$5!\,a_5 = \phi^{V}(0) = 2\phi''(0) - 3\phi''(0) + \phi''(0) - \phi^{IV}(0) = 4$$

$$a_5 = \frac{4}{5!} = \frac{1}{5\cdot 3\cdot 2} = \frac{1}{30}$$

Portanto,

$$y(x) = 1 - x + x^2 - \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{30} + O(x^6). \qquad (2.20)$$

Esta solução converge para todo x.

4. Resolva a equação do problema 3, agora com condições iniciais $y(\pi)=1$, $y'(\pi)=0$. Determine ao menos 4 termos não nulos e estime o raio de convergência.

Solução Supomos

$$\phi(x) = y(x) = \sum_{n=0}^{\infty} c_n (x-\pi)^n. \qquad (2.21)$$

Usando (2.16) temos

$$2a_2 = \phi''(\pi) = 1 - \cos\pi\, \phi'(\pi) - \pi\,\phi(\pi)$$
$$= 1-\pi$$
$$a_2 = \frac{1-\pi}{2}$$

De (2.17),

$$3!\,a_3 = \phi'''(\pi) = -\phi(\pi) + \phi''(\pi) = -1+1-\pi = -\pi$$
$$a_3 = -\frac{\pi}{6}$$

De (2.18)

$$4!\,a_4 = \phi^{iv}(\pi) = -\pi\,\phi''(\pi) + \phi'''(\pi)$$
$$= (-1+\pi)\pi - \pi = -2\pi+\pi^2$$
$$a_4 = \frac{-2\pi+\pi^2}{4\cdot 3\cdot 2} = \frac{\pi^2-2\pi}{24}.$$

Portanto,

$$y(x) = 1 + \frac{1-\pi}{2}(x-\pi)^2 - \frac{\pi}{6}(x-\pi)^3 + \frac{\pi^2-2\pi}{24}(x-\pi)^4 + O(x^5) \qquad (2.22)$$

Este série converge para todo x.