Cálculo IV

Prova 4 - **Séries de Fourier**

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1 Determine a série de Fourier da função f de período 2π definida por

$$f(t) = \begin{cases} -1 & -\pi \le t \textbf{ and } t < 0 \\ 1 & 0 \le t \textbf{ and } t < \pi \end{cases}$$

Para onde esta série converge quando $t \to 0$?

Solucão.

Consideremos uma expansão da forma

$$f(x) = c + \sum_{k=1}^{\infty} \left(a_k \cos k\, x + b_k \sin k\, x \right)$$

Os coeficientes c e a_k são zero, pois a função e ímpar. Calculemos a componente b_k :

```
> restart;
> f := t->piecewise(`and`(-Pi <= t, t < 0), -1, `and`(0 <= t, t < Pi), 1);
```

$$f := t \to piecewise\left(and\left(-\pi \le t, t < 0\right), -1, and\left(0 \le t, t < \pi\right), 1\right) \quad \textbf{(1.1)}$$

```
> assume(k, integer);
> b:=1/Pi*int('f(t)'*sin(k*t),t=-Pi..Pi);
```

$$b := -\frac{2\left((-1)^{k\sim} - 1\right)}{\pi\, k\sim} \quad \textbf{(1.2)}$$

```
> b:=subs(k=kk,b);
```

$$b := -\frac{2\left((-1)^{kk} - 1\right)}{\pi\, kk} \quad \textbf{(1.3)}$$

```
> b := unapply(%,kk);
```

$$b := kk \to -\frac{2\left((-1)^{kk} - 1\right)}{\pi\, kk} \quad \textbf{(1.4)}$$

A série de Fourier é então dada por

```
> FS := (x,n) -> Sum(b(k)*sin(k*x),k=1..n);
```

$$FS := (x, n) \to \sum_{k=1}^{n} b(k)\, \sin(k\, x) \quad \textbf{(1.5)}$$

```
> FS(t, n);
```

$$\sum_{k\sim=1}^{n}\left(-\frac{2\left(\left(-1\right)^{k\sim}-1\right)\sin\left(k\sim t\right)}{\pi\, k\sim}\right) \qquad \textbf{(1.6)}$$

Os primeiros termos da série são

```
> value(FS(t, 6));
```

$$\frac{4\sin(t)}{\pi}+\frac{4}{3}\,\frac{\sin(3\,t)}{\pi}+\frac{4}{5}\,\frac{\sin(5\,t)}{\pi} \qquad \textbf{(1.7)}$$

Em $t=0$ a série converge para 0.

2 Determine a corrente de estado estacionário de um circuito RLC em série, com $R=50\ ohms$, $L=10\ H$, $C=10^{-4}F$, com impulso externo $E(t)=50\,t\left(\pi^2-t^2\right)$ para $-\pi<t<\pi$ e $E\left(t+2\,\pi\right)=E(t)$. Encontre ao menos três termos da série para $i(t)$.

Solução.

A equação diferencial para este sistema é dada por

```
> restart;
> E:=t-> 50*t*(Pi^2-t^2);
```

$$E:=t\rightarrow 50\,t\left(\pi^2-t^2\right) \qquad \textbf{(2.1)}$$

```
> eq:=L*diff(i(t),t$2)+R*diff(i(t),t)+i(t)/C=diff(En(t), t);
```

$$eq:=L\left(\frac{d^2}{dt^2}\,i(t)\right)+R\left(\frac{d}{dt}\,i(t)\right)+\frac{i(t)}{C}=\frac{d}{dt}\,En(t) \qquad \textbf{(2.2)}$$

```
>
```

Inicialmente desenvolvemos $E(t)$ numa série de Fourier:
Como a função é ímpar, $c=0$ e $a_k=0$.

```
> assume(k, integer);
> b:=int(E(t)*sin(k*t), t = - Pi .. Pi)/Pi;
```

$$b:=\frac{600\,(-1)^{1+k\sim}}{k\sim^3} \qquad \textbf{(2.3)}$$

$> b := subs(k=kk, b);$

$$b:=\frac{600\,(-1)^{1+kk}}{kk^3} \qquad \textbf{(2.4)}$$

$> b := unapply(\%, kk);$

$$b:=kk\rightarrow\frac{600\,(-1)^{kk+1}}{kk^3} \qquad \textbf{(2.5)}$$

$> ES := (t, n) \rightarrow Sum(b(k) * \sin(k * t), k=1..n);$

$$ES:=(t,n)\rightarrow\sum_{k=1}^{n}b(k)\,\sin(k\,t) \qquad \textbf{(2.6)}$$

Os primeiros termos da série são:

$> value(ES(t, 7))$

$$600\sin(t)-75\sin(2\,t)+\frac{200}{9}\sin(3\,t)-\frac{75}{8}\sin(4\,t)+\frac{24}{5}\sin(5\,t)-\frac{25}{9}\sin(6\,t) \qquad \textbf{(2.7)}$$

$$+\frac{600}{343}\sin(7\,t)$$

A solução de estado estacionário é da forma

> $i := (t) \rightarrow A\cdot\cos(n\cdot t) + B\cdot\sin(n\cdot t)$

$$i := t \rightarrow A\cos(n\,t) + B\sin(n\,t) \quad (2.8)$$

> $En := t \rightarrow b_n\cdot\sin(n\cdot t)$

$$En := t \rightarrow b_n \sin(n\,t) \quad (2.9)$$

> eq

$$L\left(-A\cos(n\,t)\,n^2 - B\sin(n\,t)\,n^2\right) + R\left(-A\sin(n\,t)\,n + B\cos(n\,t)\,n\right) + \frac{A\cos(n\,t) + B\sin(n\,t)}{C} = b_n\cos(n\,t)\,n \quad (2.10)$$

> $eq1 := expand(lhs(eq) - rhs(eq))$

$$eq1 := -L\,A\cos(n\,t)\,n^2 - L\,B\sin(n\,t)\,n^2 - R\,A\sin(n\,t)\,n + R\,B\cos(n\,t)\,n + \frac{A\cos(n\,t)}{C} + \frac{B\sin(n\,t)}{C} - b_n\cos(n\,t)\,n \quad (2.11)$$

> $eq2 := collect(eq1, \cos(n\cdot t))$

$$eq2 := \left(-L\,A\,n^2 + R\,B\,n + \frac{A}{C} - b_n\,n\right)\cos(n\,t) - L\,B\sin(n\,t)\,n^2 - R\,A\sin(n\,t)\,n + \frac{B\sin(n\,t)}{C} \quad (2.12)$$

> $eq3 := collect(eq2, \sin(n\cdot t))$

$$eq3 := \left(-L\,B\,n^2 - R\,A\,n + \frac{B}{C}\right)\sin(n\,t) + \left(-L\,A\,n^2 + R\,B\,n + \frac{A}{C} - b_n\,n\right)\cos(n\,t) \quad (2.13)$$

> $ee1 := coeff(eq3, \sin(n\cdot t)) = 0$

$$ee1 := -L\,B\,n^2 - R\,A\,n + \frac{B}{C} = 0 \quad (2.14)$$

> $ee2 := coeff(eq3, \cos(n\cdot t)) = 0$

$$ee2 := -L\,A\,n^2 + R\,B\,n + \frac{A}{C} - b_n\,n = 0 \quad (2.15)$$

> $sol := solve(\{ee1, ee2\}, \{A, B\})$

$$sol := \left\{A = -\frac{b_n\,n\,C\,(L\,n^2\,C - 1)}{L^2\,n^4\,C^2 - 2\,L\,n^2\,C + R^2\,n^2\,C^2 + 1},\ B = \frac{b_n\,n^2\,C^2\,R}{L^2\,n^4\,C^2 - 2\,L\,n^2\,C + R^2\,n^2\,C^2 + 1}\right\} \quad (2.16)$$

> $assign(sol)$

> $A := subs\left(b_n = b(n), A\right)$

$$A := -\frac{600\,(-1)^{n+1}\,C\,(L\,n^2\,C - 1)}{n^2\,(L^2\,n^4\,C^2 - 2\,L\,n^2\,C + R^2\,n^2\,C^2 + 1)} \quad (2.17)$$

> $B := subs\left(b_n = b(n), B\right)$

$$B := \frac{600\,(-1)^{n+1}\,C^2\,R}{n\,(L^2\,n^4\,C^2 - 2\,L\,n^2\,C + R^2\,n^2\,C^2 + 1)} \quad (2.18)$$

> $A := unapply(A, n)$

$$A := n \rightarrow -\frac{600\,(-1)^{n+1}\,C\,(L\,n^2\,C-1)}{n^2\,(L^2\,n^4\,C^2-2\,L\,n^2\,C+R^2\,n^2\,C^2+1)} \qquad \textbf{(2.19)}$$

> $B := unapply(B, n)$

$$B := n \rightarrow \frac{600\,(-1)^{n+1}\,C^2\,R}{n\,(L^2\,n^4\,C^2-2\,L\,n^2\,C+R^2\,n^2\,C^2+1)} \qquad \textbf{(2.20)}$$

>

As soluções são da forma:

> $i := Sum(A(j)\cdot\cos(j\cdot t) + B(j)\cdot\sin(j\cdot t), j = 1..n)$

$$i := \sum_{j=1}^{n}\left(-\frac{600\,(-1)^{j+1}\,C\,(L\,j^2\,C-1)\cos(j\,t)}{j^2\,(L^2\,j^4\,C^2-2\,L\,j^2\,C+R^2\,j^2\,C^2+1)} + \frac{600\,(-1)^{j+1}\,C^2\,R\sin(j\,t)}{j\,(L^2\,j^4\,C^2-2\,L\,j^2\,C+R^2\,j^2\,C^2+1)}\right) \qquad \textbf{(2.21)}$$

Utilizando três termos e os valores de C, L e R dados no problema temos

> $value(subs(n = 3, C = 10^{-4}, L = 10, R = 50, i))$

$$\frac{29970}{499013}\cos(t) + \frac{150}{499013}\sin(t) - \frac{3735}{248029}\cos(2\,t) - \frac{75}{496058}\sin(2\,t) + \frac{9910}{1473459}\cos(3\,t) + \frac{50}{491153}\sin(3\,t) \qquad \textbf{(2.22)}$$

> $evalf(\%)$

$$0.06005855559\cos(t) + 0.0003005933713\sin(t) - 0.01505872297\cos(2.\,t) - 0.0001511919977\sin(2.\,t) + 0.006725670684\cos(3.\,t) + 0.0001018012717\sin(3.\,t) \qquad \textbf{(2.23)}$$

>

3 Determine a série de Fourier da função f definida por

$$f(t) = \begin{cases} 0 & -2 < t \textbf{ and } t < -1 \\ k & -1 < t \textbf{ and } t < 1 \\ 0 & 1 < t \textbf{ and } t < 2 \end{cases},$$

com f(t + 4) = f(t).

$$c = \frac{1}{T}\int_{\alpha}^{\beta} f(x)\,dx,\ a_k = \frac{2}{T}\int_{\alpha}^{\beta} f(x)\cos\left(\frac{2\,k\,\pi\,x}{T}\right)dx,\ b_k = \frac{2}{T}\int_{\alpha}^{\beta} f(x)\sin\left(\frac{2\,k\,\pi\,x}{T}\right)dx$$

```
> restart;
> T := 4;
```

$$T := 4 \qquad \textbf{(3.1)}$$

```
> f := piecewise(`and` (-2 < t, t < -1),0,`and`(-1 < t, t < 1),
  k,`and` (1 < t, t < 2),0);
```

$$f := \begin{cases} 0 & -2 < t \textbf{ and } t < -1 \\ k & -1 < t \textbf{ and } t < 1 \\ 0 & 1 < t \textbf{ and } t < 2 \end{cases} \quad (3.2)$$

Como esta função é par, $b_k = 0$. Os outros coeficientes são dados por

```
> c := (int(f, t = -2 .. 2))/T;
```

$$c := \frac{1}{2} k \quad (3.3)$$

```
> assume(n, integer);
> a := 2*(int(f*cos(2*n*Pi*t/T), t = -2 .. 2))/T;
```

$$a := \frac{2 \sin\left(\frac{1}{2} n\sim \pi\right) k}{n\sim \pi} \quad (3.4)$$

```
> a := unapply(a, n)
```

$$a := n\sim \rightarrow \frac{2 \sin\left(\frac{1}{2} n\sim \pi\right) k}{n\sim \pi} \quad (3.5)$$

```
> FS := (t, l) → c + Sum(a(i) * cos(2 * i * π * t/T), i = 1 ..l);
```

$$FS := (t, l) \rightarrow c + \sum_{i=1}^{l} a(i) \cos\left(\frac{2 i \pi t}{T}\right) \quad (3.6)$$

Os primeiros termos da série são então dados por

```
> value(FS(t, 8))
```

$$\frac{1}{2} k + \frac{2 k \cos\left(\frac{1}{2} \pi t\right)}{\pi} - \frac{2}{3} \frac{k \cos\left(\frac{3}{2} \pi t\right)}{\pi} + \frac{2}{5} \frac{k \cos\left(\frac{5}{2} \pi t\right)}{\pi} - \frac{2}{7} \frac{k \cos\left(\frac{7}{2} \pi t\right)}{\pi} \quad (3.7)$$

4 Calcule a transformada de Fourier discreta de $(1, -i, -1, i)$ e a transformada inversa de $(1, i, -1, -1)$

Solução

```
> with(LinearAlgebra) :
> n := 4
```

$$n := 4 \quad (4.1)$$

```
> ξ := exp(-2 * π * I/n);
```

$$\xi := -I \quad (4.2)$$

```
> ω := exp(2 * π * I/n);
```

$$\omega := I \quad (4.3)$$

```
> F4 := (VandermondeMatrix([seq(ξ^i, i = 0 ..n-1)]))
```

$$F4 := \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -I & -1 & I \\ 1 & -1 & 1 & -1 \\ 1 & I & -1 & -I \end{bmatrix} \tag{4.4}$$

> $X := \langle 1, -I, -1, I \rangle$

$$X := \begin{bmatrix} 1 \\ -I \\ -1 \\ I \end{bmatrix} \tag{4.5}$$

> $F4.X$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix} \tag{4.6}$$

> $F4inv := \left(\frac{1}{n}\right) \cdot map(conjugate, F4)$

$$F4inv := \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4}\,I & -\frac{1}{4} & -\frac{1}{4}\,I \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4}\,I & -\frac{1}{4} & \frac{1}{4}\,I \end{bmatrix} \tag{4.7}$$

> $Y := \langle 1, I, -1, -1 \rangle$

$$Y := \begin{bmatrix} 1 \\ I \\ -1 \\ -1 \end{bmatrix} \tag{4.8}$$

> $F4inv.Y$

(4.9)

$$\begin{bmatrix} -\frac{1}{4}+\frac{1}{4}\,I \\ \frac{1}{4}+\frac{1}{4}\,I \\ \frac{1}{4}-\frac{1}{4}\,I \\ \frac{3}{4}-\frac{1}{4}\,I \end{bmatrix} \quad \textbf{(4.9)}$$

>