Distribuição Normal

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

densidade de probabilidade

Propriedades

```
> assume(mu,real); assume(sigma,real);assume(sigma>0);
```

```
> f := t->1/2*2^(1/2)/Pi^(1/2)/sigma*exp(-1/2*(t-mu)^2/sigma^2);
```

$$f := t \rightarrow \frac{1}{2} \frac{\sqrt{2}\, \mathbf{e}^{\left(-\frac{1}{2} \frac{(t-\mu)^2}{\sigma^2}\right)}}{\sqrt{\pi}\, \sigma}$$

```
> Int(f(t),t=-infinity..infinity)=int(f(t),t=-infinity..infinity);
```

$$\int_{-\infty}^{\infty} \frac{1}{2} \frac{\sqrt{2}\, \mathbf{e}^{\left(-\frac{(t-\mu\sim)^2}{2\, \sigma\sim^2}\right)}}{\sqrt{\pi}\, \sigma\sim}\, dt = 1$$

```
> Int(t*f(t),t=-infinity..infinity)=int(t*f(t),t=-infinity..infinity);
```

$$\int_{-\infty}^{\infty} \frac{1}{2} \frac{t\sqrt{2}\,\mathbf{e}^{\left(-\frac{(t-\mu\sim)^2}{2\,\sigma\sim^2}\right)}}{\sqrt{\pi}\,\sigma\sim}\,dt = \mu\sim$$

```
>Int((t-mu)^2*f(t),t=-infinity..infinity)=int((t-mu)^2*f(t),
t=-infinity..infinity);
```

$$\int_{-\infty}^{\infty} \frac{1}{2} \frac{(t-\mu\sim)^2\sqrt{2}\,\mathbf{e}^{\left(-\frac{(t-\mu\sim)^2}{2\,\sigma\sim^2}\right)}}{\sqrt{\pi}\,\sigma\sim}\,dt = \sigma\sim^2$$

Exemplo. Suponha que a corrente em um pedaço de fio segue uma distribuição normal com uma média

$$\mu = 10\,mA$$

e uma variância

$$\sigma^2 = 4\,mA^2$$

Qual a probabilidade de que uma medida exceda 13 mA ?

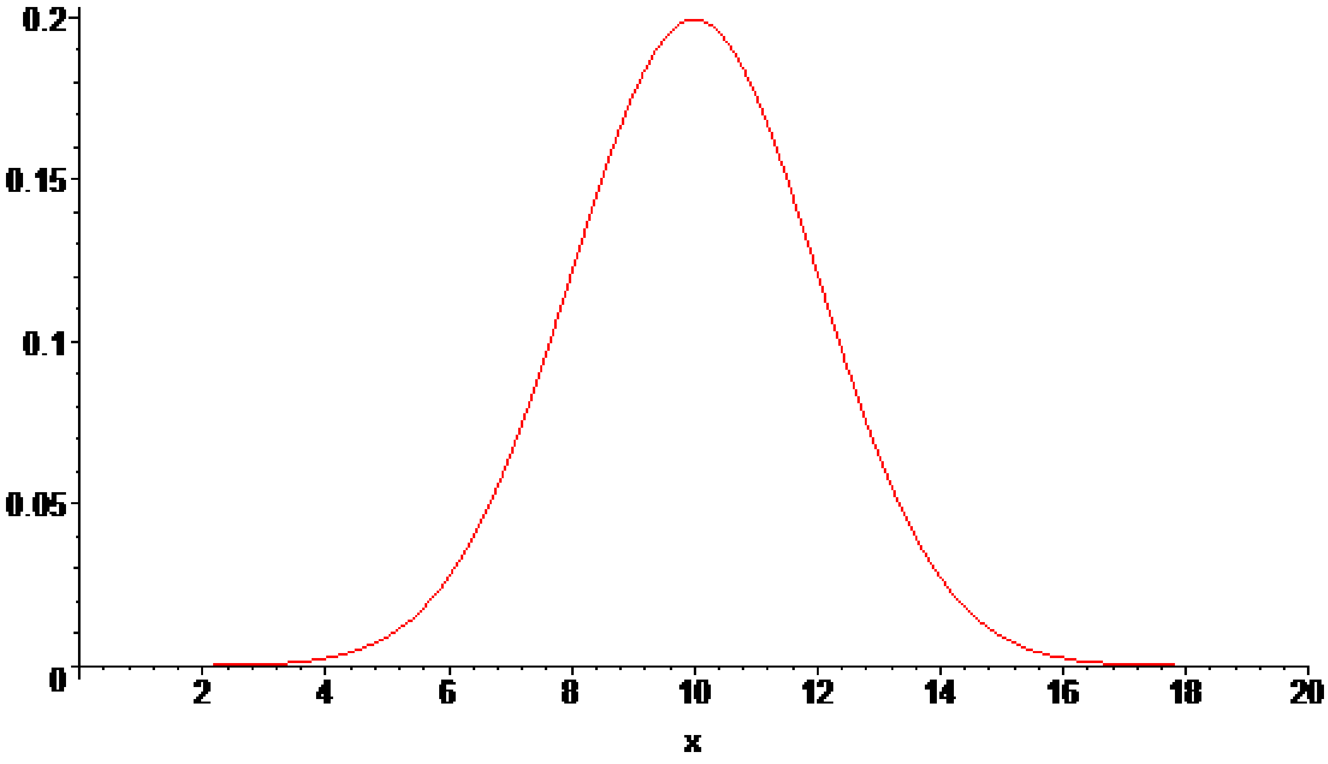
X – corrente em mA

$$P(X > 13) = ?$$

```
> f:=(mu,sigma)->1/(sqrt(2*Pi)*sigma)*exp(-(x-mu)^2/(2*sigma^2));
```

$$f := (\mu, \sigma) \to \frac{e^{\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right)}}{\sqrt{2\pi}\,\sigma}$$

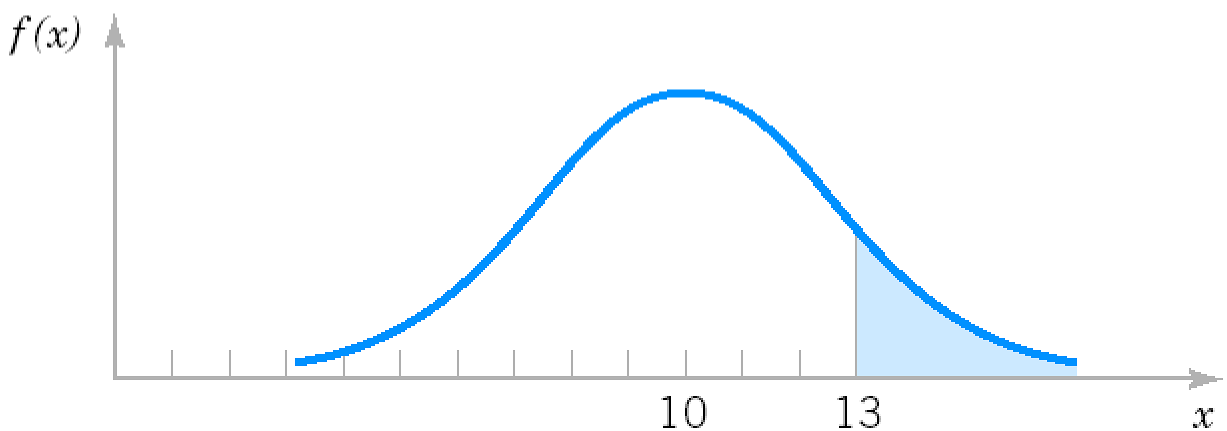
```
> plot(f(10,2),x=0..20);
```



$$P(x > 13) = \int_{13}^{\infty} \frac{1}{(10)\sqrt{2\pi}} e^{\frac{-(x-10)^2}{2(2)^2}} dx$$

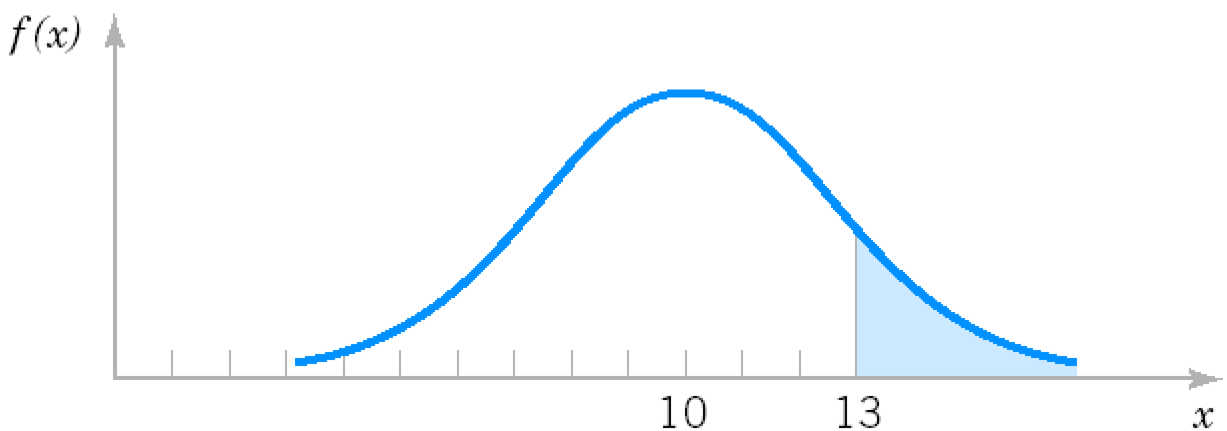
```
> Int(f(10,2),x=13..infinity)=evalf(int(f(10,2),x=13..infinity));
```

$$\int_{13}^{\infty} \frac{\frac{1}{4}\sqrt{2}\, \mathrm{e}^{\left(-\frac{(x-10)^2}{8}\right)}}{\sqrt{\pi}} dx = 0.06680720$$



```
> Int(f(10,2),x=13..infinity)=evalf(int(f(10,2),x=13..infinity));
```

$$\int_{13}^{\infty} \frac{1}{4}\,\frac{\sqrt{2}\,\mathbf{e}^{\left(-\frac{(x-10)^2}{8}\right)}}{\sqrt{\pi}}\,dx = 0.06680720$$



Cálculo automático no Maple

μ σ

```
> with(stats):
> 1-statevalf[cdf,normald[10,2]](13.0);
```

0.0668072

x

cumulative density function

Notemos que

$$P(\mu - \sigma < X < \mu + \sigma) = 0.6826894920$$
$$P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9544997360$$
$$P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973002039$$

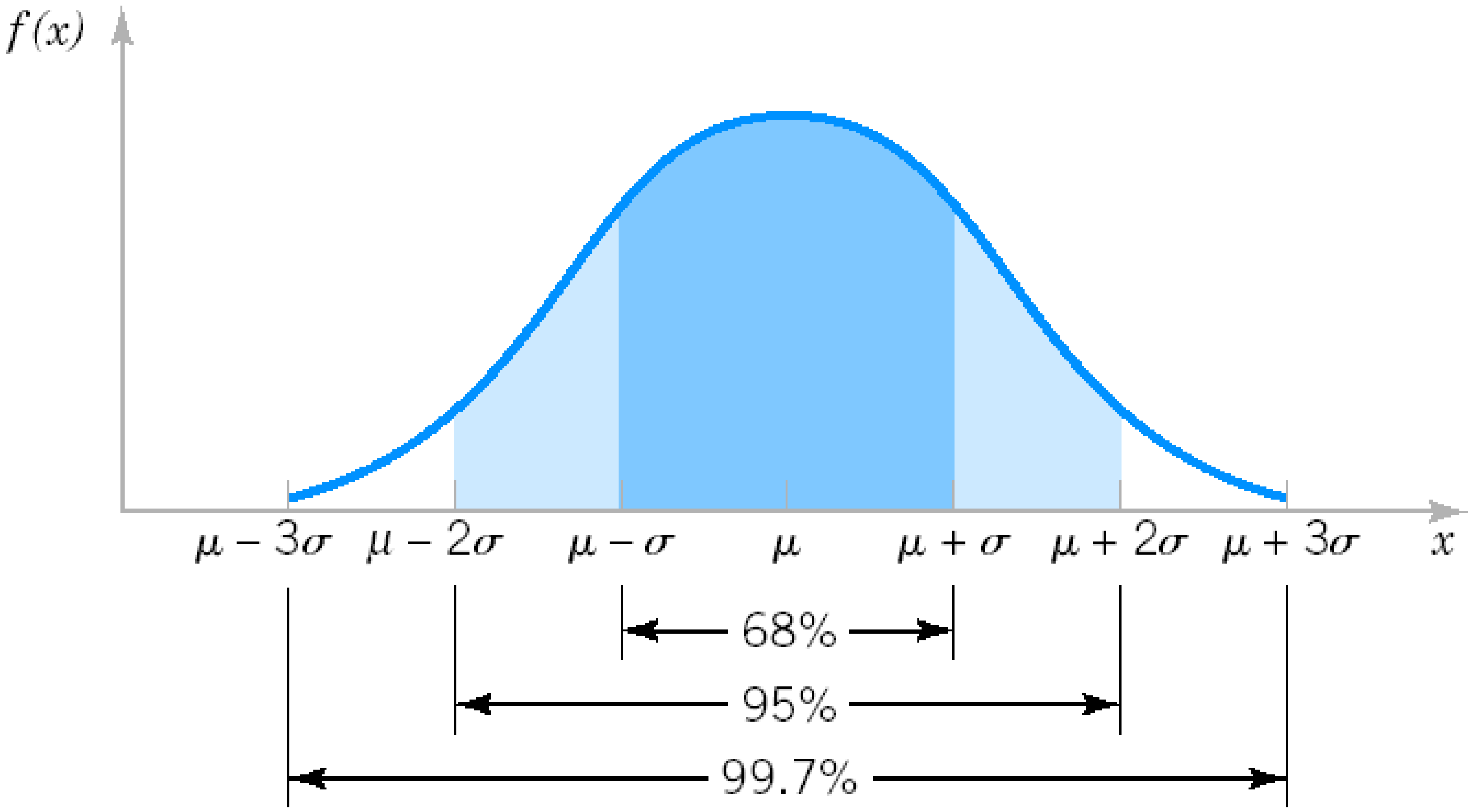
Prova:

```
> for n to 3 do
    Int(f(mu,sigma,n),x=mu-n*sigma..mu+n*sigma)
 =evalf(int(f(mu,sigma,n),x=mu-n*sigma..mu+n*sigma));
  od;
```

$$\int_{\mu-\sigma}^{\mu+\sigma} \frac{1}{2} \cdot \frac{\sqrt{2}\,\mathbf{e}^{\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}}{\sqrt{\pi}\,\sigma}\,dx = 0.682689492$$

$$\int_{\mu-3\sigma}^{\mu+3\sigma} \frac{1}{2} \cdot \frac{\sqrt{2}\,\mathbf{e}^{\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}}{\sqrt{\pi}\,\sigma}\,dx = 0.99730020$$

$$\int_{\mu-2\sigma}^{\mu+2\sigma} \frac{1}{2} \cdot \frac{\sqrt{2}\,\mathbf{e}^{\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}}{\sqrt{\pi}\,\sigma}\,dx = 0.95449973$$



f(x)
μ − 3σ
μ − 2σ
μ − σ
μ
μ + σ
μ + 2σ
μ + 3σ
x
68%
95%
99.7%

Por simetria,

$$P(x > \mu) = P(x < \mu) = 0.5$$

```
>Int(f(mu,sigma),x=-infinity..infinity)=
>evalf(int(f(mu,sigma),x=-infinity..infinity));
```

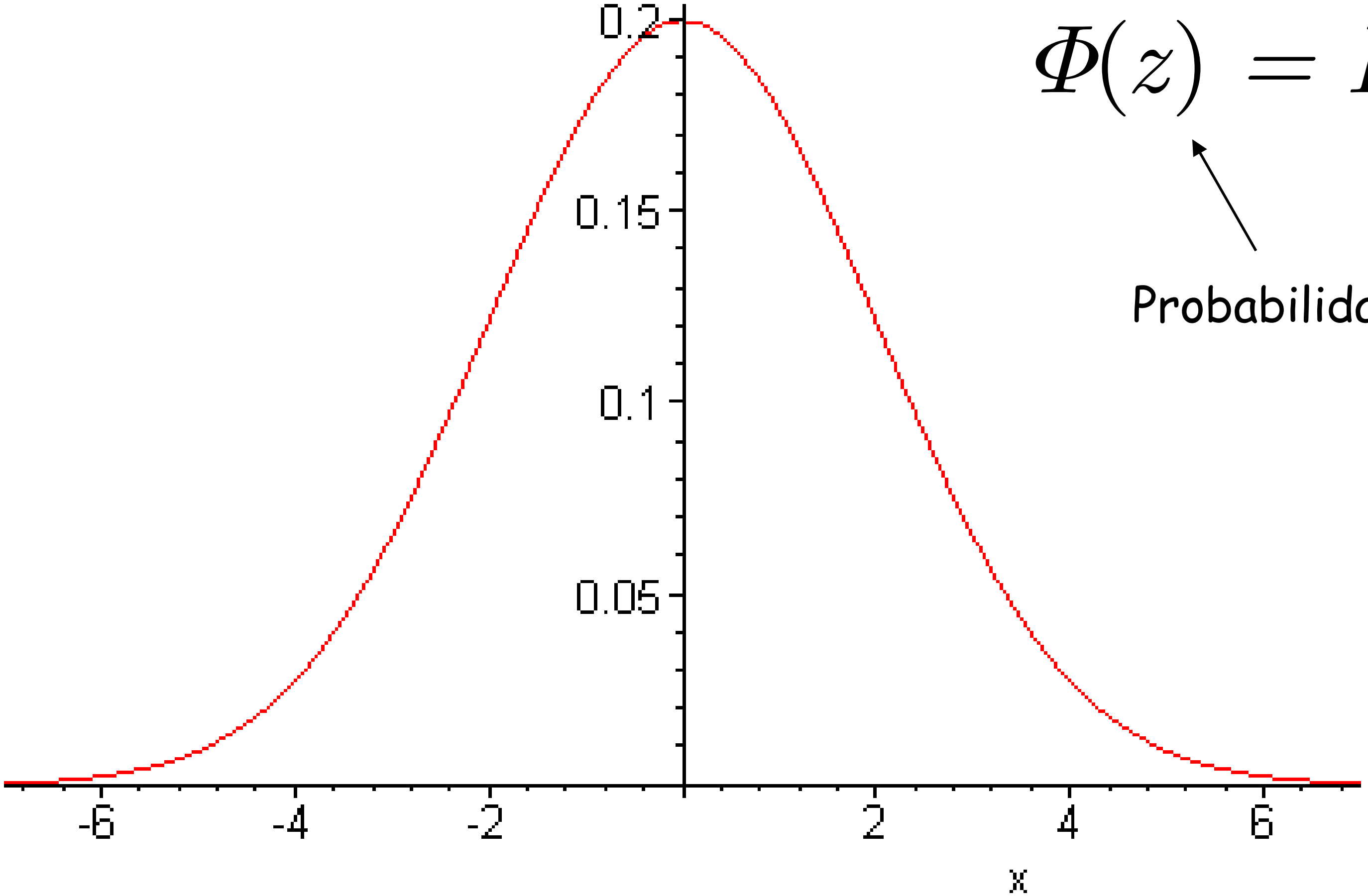
$$\int_{-\infty}^{\infty} \frac{1}{2} \frac{\sqrt{2}\, \mathbf{e}^{\left(-\frac{(x-\mu)^2}{2\,\sigma\sim^2}\right)}}{\sqrt{\pi}\,\sigma\sim}\, dx = 1.$$

Largura da curva normal: 6σ

$$P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973002039$$

Variável aleatória normal padrão: Z

$$\mu = 0\,, \quad \sigma^2 = 1$$

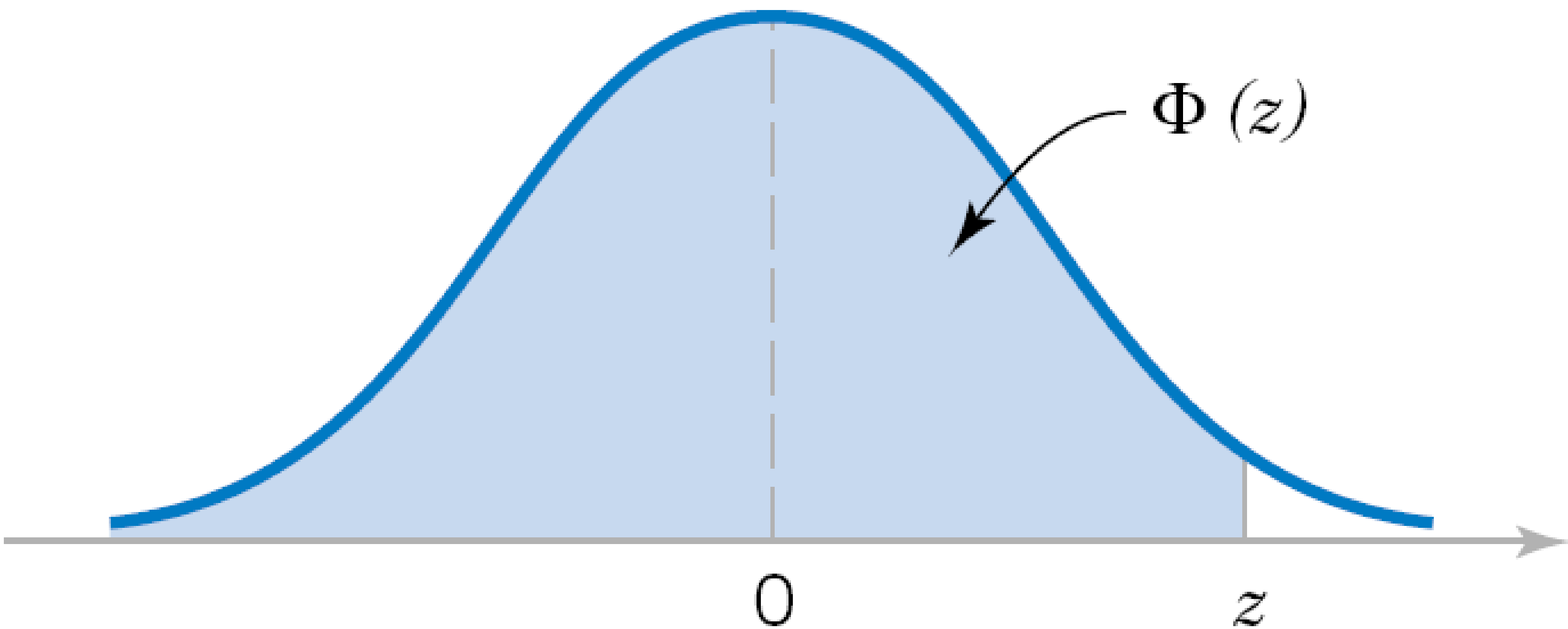


$$\Phi(z) = P(Z \leq z)$$

Probabilidade cumulativa

Distribuição Normal Padrão Cumulativa

$$\Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du$$



$$\Phi(z) = P(Z \leq z)$$

Cálculo usando Maple

```
> Phi:=z->int(f(0,1),x=-infinity..z);
```

$$\Phi := z \to \int_{-\infty}^{z} f(0, 1)\, dx$$

```
> evalf(Phi(1.57));
```

$$0.9417924$$

Ou, usando o pacote Statistics do Maple,

```
> with(Statistics);
> mu := 0; sigma := 1;
> X := RandomVariable(Normal(mu, sigma));;
> CDF(X, 1.57);
```

0.94179244361447045

Usando tabela

Table II Cumulative Standard Normal Distribution (*continued*)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.500000	0.503989	0.507978	0.511967	0.515953	0.519939	0.532922	0.527903	0.531881	0.535856
0.1	0.539828	0.543795	0.547758	0.551717	0.555760	0.559618	0.563559	0.567495	0.571424	0.575345
0.2	0.579260	0.583166	0.587064	0.590954	0.594835	0.598706	0.602568	0.606420	0.610261	0.614092
0.3	0.617911	0.621719	0.625516	0.629300	0.633072	0.636831	0.640576	0.644309	0.648027	0.651732
0.4	0.655422	0.659097	0.662757	0.666402	0.670031	0.673645	0.677242	0.680822	0.684386	0.687933
0.5	0.691462	0.694974	0.698468	0.701944	0.705401	0.708840	0.712260	0.715661	0.719043	0.722405
0.6	0.725747	0.729069	0.732371	0.735653	0.738914	0.742154	0.745373	0.748571	0.751748	0.754903
0.7	0.758036	0.761148	0.764238	0.767305	0.770350	0.773373	0.776373	0.779350	0.782305	0.785236
0.8	0.788145	0.791030	0.793892	0.796731	0.799546	0.802338	0.805106	0.807850	0.810570	0.813267
0.9	0.815940	0.818589	0.821214	0.823815	0.826391	0.828944	0.831472	0.833977	0.836457	0.838913
1.0	0.841345	0.843752	0.846136	0.848495	0.850830	0.853141	0.855428	0.857690	0.859929	0.862143
1.1	0.864334	0.866500	0.868643	0.870762	0.872857	0.874928	0.876976	0.878999	0.881000	0.882977
1.2	0.884930	0.886860	0.888767	0.890651	0.892512	0.894350	0.896165	0.897958	0.899727	0.901475
1.3	0.903199	0.904902	0.906582	0.908241	0.909877	0.911492	0.913085	0.914657	0.916207	0.917736
1.4	0.919243	0.920730	0.922196	0.923641	0.925066	0.926471	0.927855	0.929219	0.930563	0.931888
1.5	0.933193	0.934478	0.935744	0.936992	0.938220	0.939429	0.940620	0.941792	0.942947	0.944083
1.6	0.945201	0.946301	0.947384	0.948449	0.949497	0.950529	0.951543	0.952540	0.953521	0.954486
1.7	0.955435	0.956367	0.957284	0.958185	0.959071	0.959941	0.960796	0.961636	0.962462	0.963273

Padronização

Se X é uma variável aleatória normal com

$$E(X) = \mu \qquad V(X) = \sigma^2$$

Então

$$Z = \frac{X - \mu}{\sigma}$$

Variável aleatória normal padrão

$$E(Z) = 0$$

$$V(Z) = 1$$

Seja X uma variável aleatória normal, com

$$E(X) = \mu \qquad\qquad V(X) = \sigma^2$$

Então

$$P(X \leq x) = P\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right) = P(Z \leq z)$$

$$z = \frac{x-\mu}{\sigma}$$

valor z obtido por padronização de X

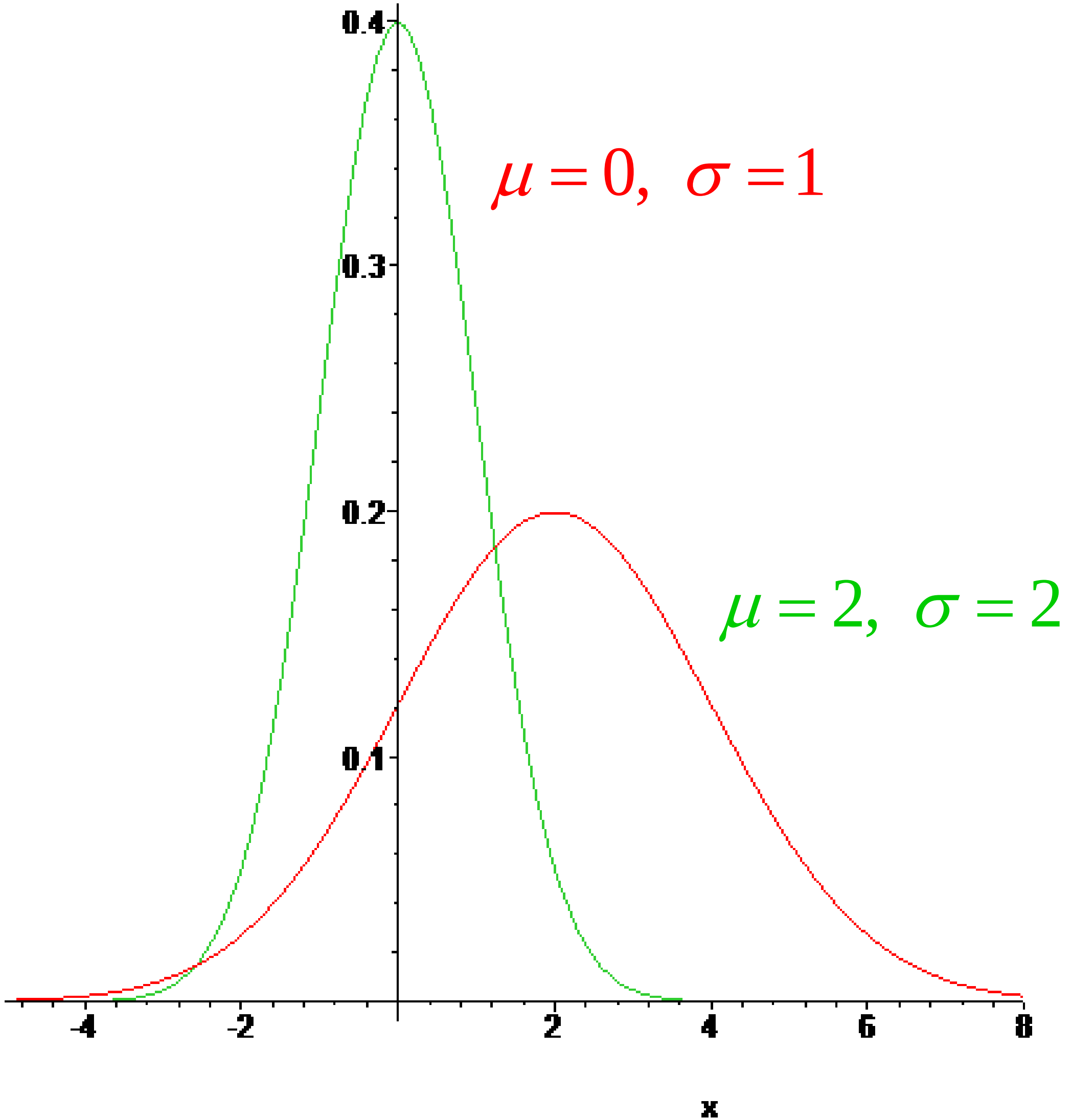
Gráficos de distribuições no Maple usando o pacote stats:

```
> with(stats):

> normpdf:=(x,mu,sigma)->statevalf[pdf,normald[mu,sigma]](x);
```

$$normpdf := (x, \mu, \sigma) \rightarrow statevalf_{pdf,\, normald_{\mu,\sigma}}(x)$$

```
> plot([normpdf(x,2,2),normpdf(x,0,1)],x=-5..8);
```



0.4
0.3
0.2
0.1
-4
-2
2
4
6
8
x
$\mu = 0, \ \sigma = 1$
$\mu = 2, \ \sigma = 2$

Examplo. Seja X uma variável aletória normal com

$$\mu = 10 \qquad \sigma = 2$$

Determine

$$P(9 < X < 11)$$

$$P(9 < X < 11) = P\left(\frac{9-10}{2} < \frac{X-10}{2} < \frac{11-10}{2}\right)$$

$$= P\left(-0.5 < Z < 0.5\right)$$

$$= P(Z < 0.5) - P(Z < -0.5)$$

z	0.00	0.01
0.0	0.500000	0.503989
0.1	0.539828	0.543795
0.2	0.579260	0.583166
0.3	0.617911	0.621719
0.4	0.655422	0.659097
0.5	0.691462	0.694974
0.6	0.725747	0.729069

```
> evalf(int(f(10,2),x=9..11));
```

0.3829249

−0.5	0.277595	0.280957	0.284339	0.287740	0.291160	0.294599	0.298056	0.301532	0.305026	0.308538
−0.4	0.312067	0.315614	0.319178	0.322758	0.326355	0.329969	0.333598	0.337243	0.340903	0.344578
−0.3	0.348268	0.351973	0.355691	0.359424	0.363169	0.366928	0.370700	0.374484	0.378281	0.382089
−0.2	0.385908	0.389739	0.393580	0.397432	0.401294	0.405165	0.409046	0.412936	0.416834	0.420740
−0.1	0.424655	0.428576	0.432505	0.436441	0.440382	0.444330	0.448283	0.452242	0.456205	0.460172
0.0	0.464144	0.468119	0.472097	0.476078	0.480061	0.484047	0.488033	0.492022	0.496011	0.500000

$$P(9 < X < 11) = P(Z < 0.5) - P(Z < -0.5)$$
$$= 0.691462 - 0.308528$$
$$= 0.382934$$

Determine x tal que

$$P(X < x) = 0.98$$

$$P(X < x) = P\left(\frac{X - 10}{2} < \frac{x - 10}{2} \right)$$

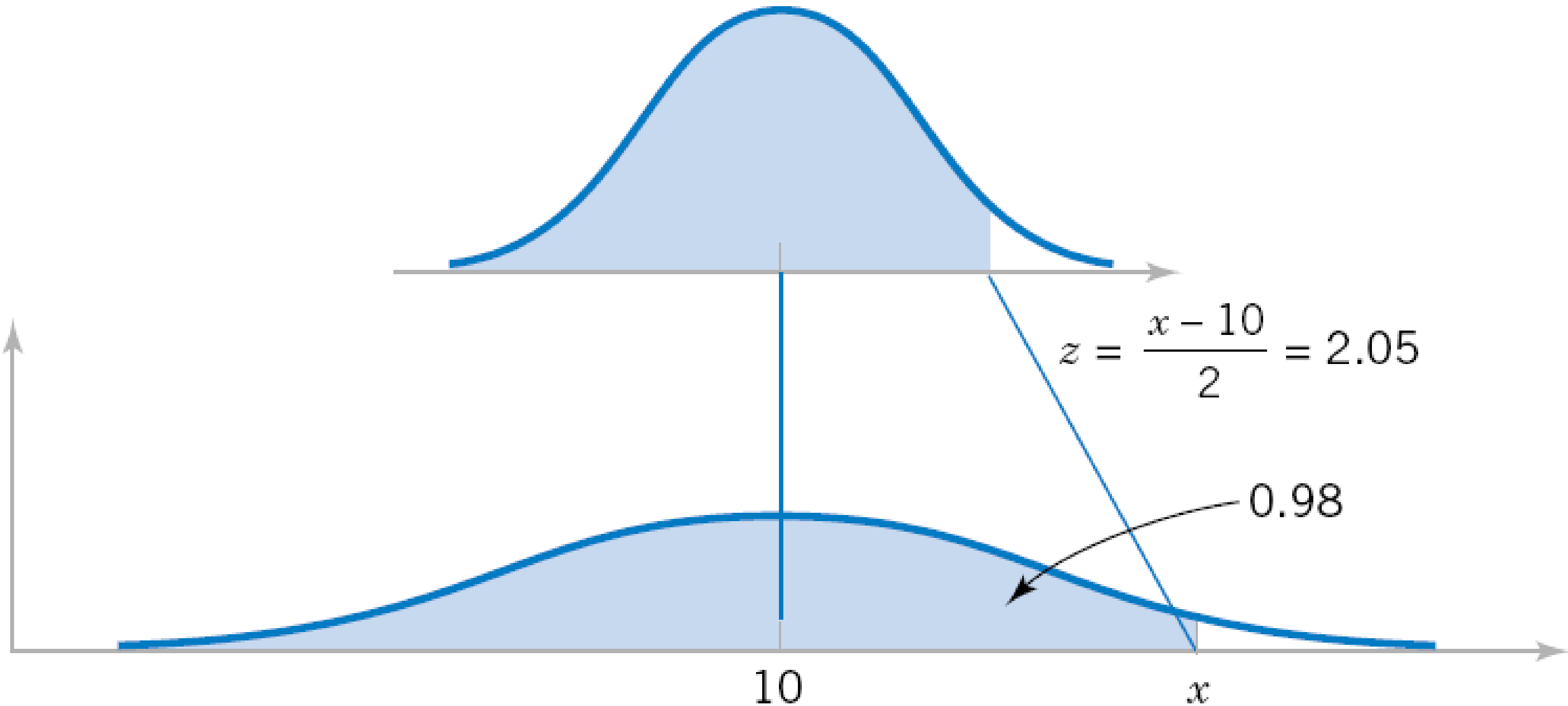
$$= P\left(Z < \frac{x - 10}{2} \right) = 0.98$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06
1.9	0.971283	0.971933	0.972571	0.973197	0.973810	0.974412	0.975002
2.0	0.977250	0.977784	0.978308	0.978822	0.979325	0.979818	0.980301
2.1	0.982136	0.982571	0.982997	0.983414	0.983823	0.984222	0.984614

Z = 2.05 é uma boa aproximação.

$$P(Z < 2.05) \simeq 0.98$$

$$\frac{x - 10}{2} = 2.05 \quad \Longrightarrow \quad x = 2(2.05) + 10 = 14.1$$



$z = \frac{x - 10}{2} = 2.05$
0.98
10
x

Procedimento em Maple

```
> mu := 10: sigma := 2:
> X := RandomVariable(Normal(mu, sigma)):
> Quantile(X,0.98);
```

```
14.10749782
```

Exemplo. Suponha que na detecção de um sinal digital, o ruído de fundo segue uma distribuição normal com uma média de 0 V e desvio padrão de 0.45 V. Neste sistema, um bit 1 é transmitido quando a tensão excede 0.9 V.

Qual é a probabilidade de detecção de um bit 1 devido ao ruído ?

X – variável aleatória normal que representa a voltagem do ruído de fundo.

$$P(X > 0.9) = P\left(\frac{X-0}{0.45} > \frac{0.9-0}{0.45}\right) = P(Z > 2)$$

$$= 1 - P(Z < 2) = 1 - 0.97725 = 0.02275$$

Probabilidade de falsa detecção

1.9	0.971283
2.0	0.977250
2.1	0.982136

Cálculo no Maple

```
> int(f(0,0.45),x=0.9..infinity);
```

0.0227501

Exemplo: Determine os limitantes simétricos em torno de 0 que incluem 99% de todas as leituras de ruído.

$$P(-x < X < x) = 0.99 = ?$$

$$P(-x < X < x) = P\left(\frac{-x}{0.45} < \frac{X}{0.45} < \frac{x}{0.45}\right)$$

$$= P\left(\frac{-x}{0.45} < Z < \frac{x}{0.45}\right) = 0.99$$

$$P(-z < Z < z) = 1 - 2P(Z < -z) = 0.99$$

$$P(Z < -z) = \frac{1 - 0.99}{2} = 0.005$$

z	-0.09	-0.08	-0.07
-2.6	0.003573	0.003681	0.003793
-2.5	0.004799	0.004940	0.005085
-2.4	0.006387	0.006569	0.006756

$$\Rightarrow z = -2.58$$

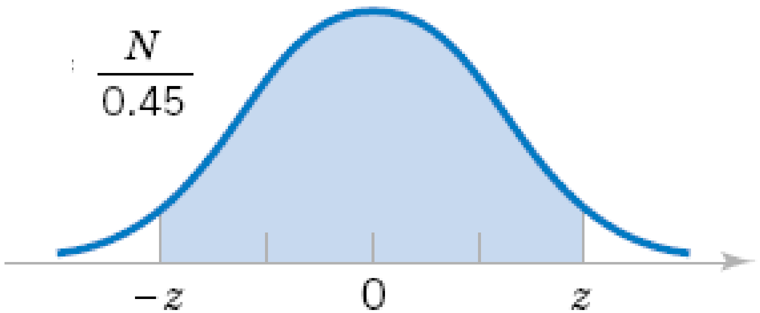
$$\frac{x}{0.45} = z = -2.58$$

$$x = -1.16$$

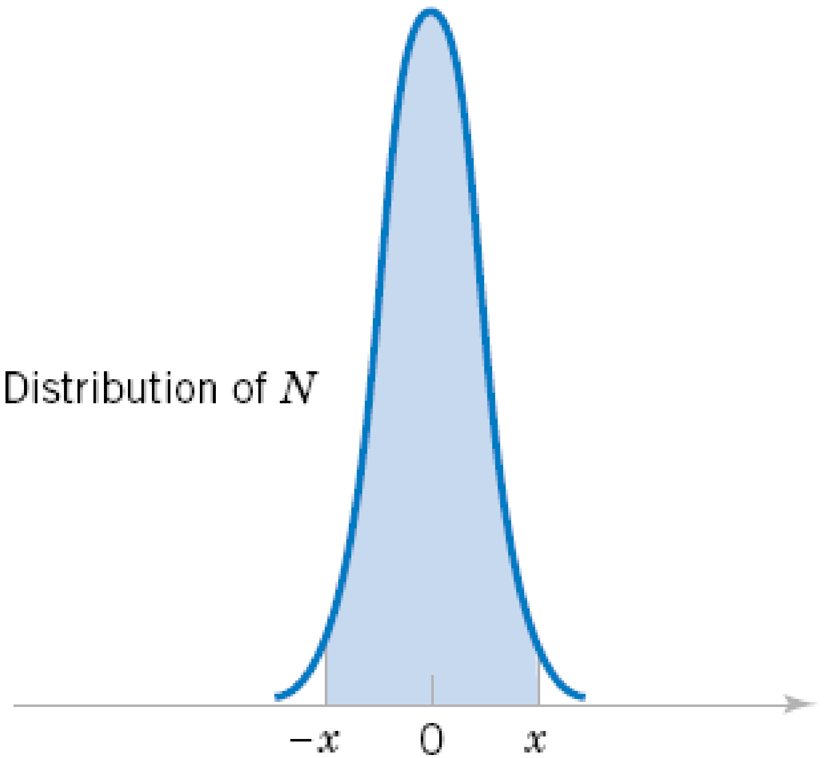
Procedimento em Maple

```
> mu := 0: sigma := 0.45:
> X := RandomVariable(Normal(mu, sigma)):
> Quantile(X,0.005);
```

-1.159123187



$$z = 2.58$$



$$x = 1.16$$

O diâmetro de um orifício em um DVD tem distribuição normal com média 0.2508 in. e desvio padrão 0.0005 in. As especificações do orifício são

$$0.2500 \pm 0.0015$$

Qual é a proporção de orifícios satisfazendo as especificações ?

X – diâmetro do orifício

$$P(0.2485 < X < 0.2515) = ?$$

$$P(0.2485 < X < 0.2515) = P\left(\frac{0.2485 - 0.2508}{0.0005} < Z < \frac{0.2515 - 0.2508}{0.0005}\right)$$

$$= P(-4.6 < Z < 1.4) = P(Z < 1.4) - P(Z < -4.6)$$

$$= 0.91924 - 0$$

Tabela marca até z = 3.9

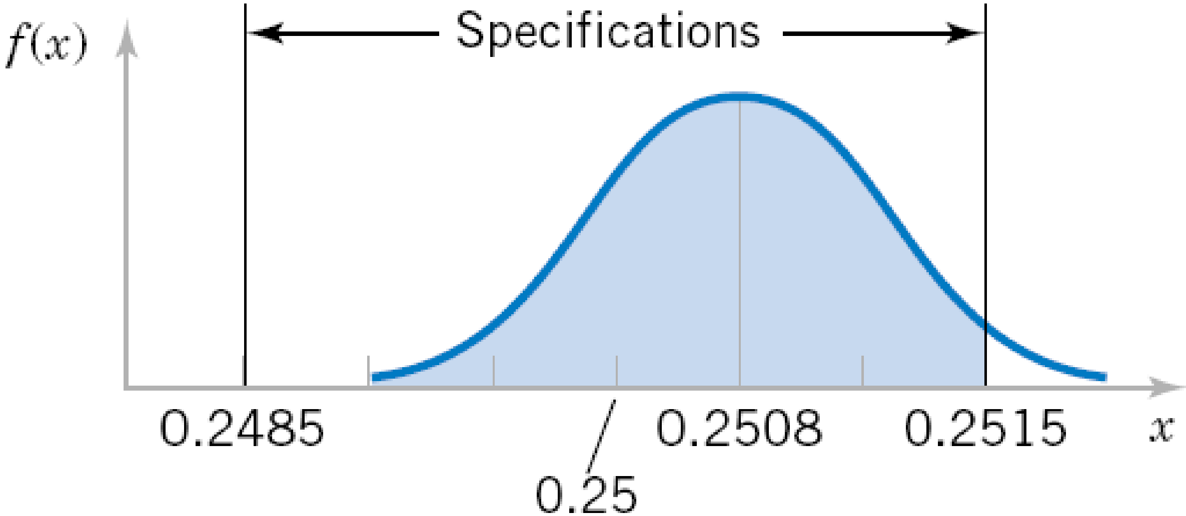
z	0.00	0.01
1.3	0.903199	0.904902
1.4	0.919243	0.920730
1.5	0.933193	0.934478

A maior parte das peças fora das especificações têm orifício muito grande

Processo deve ser centrado em 0.25

```
> mu := 0.2508: sigma := 0.0005:
> X := RandomVariable(Normal(mu, sigma)):
> CDF(X,0.2515)-CDF(X,0.2485);
```

0.9192412283



$$P(0.2485 < X < 0.2515) = P\left(\frac{0.2485-0.25}{0.0005} < Z < \frac{0.2515-0.25}{0.0005}\right)$$

$$= P(-3 < Z < 3) = P(Z < 3) - P(Z < -3)$$

$$= 1 - 2(1 - 0.99865) = \boxed{0.9973}$$

z	0.00	0.01
2.9	0.998134	0.998193
3.0	0.998650	0.998694
3.1	0.999032	0.999065

```
> mu := 0.25: sigma := 0.0005:
> X := RandomVariable(Normal(mu, sigma)):
> CDF(X,0.2515)-CDF(X,0.2485);
                    0.9973002040
```