Soluções de EDOs em Séries de Potências

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Equação de Legendre

Consideremos a equação de Legendre

$$\left(1-x^2\right)\frac{d^2}{dx^2}y(x)-2x\frac{d}{dx}y(x)+l(l+1)y(x)=0\,.$$

Busquemos uma solução em série de potências em torno do ponto ordinário $x = 0$.

```
> restart;
> y := sum(a[i]*x^i, i = k-2 .. k+2);
```

$$y := a_{k-2}x^{k-2}+a_{-1+k}x^{-1+k}+a_kx^k+a_{1+k}x^{1+k}+a_{k+2}x^{k+2} \quad (1.1)$$

```
> eq1:=(1-x^2)*(diff(y, x, x))-2*x*(diff(y,x))+l*(l+1)*y = 0:
> eq2 := simplify(eq1);
```

$$\begin{aligned} eq2 := &-a_k x^k k^2-2a_{k-2}x^{k-2}-2a_{1+k}x^{1+k}-6a_{k+2}x^{k+2}+a_kx^{k-2}k^2\\ &-a_kx^{k-2}k-x^{k-2}a_{k-2}k^2-x^{-1+k}a_{-1+k}k^2-3x^{1+k}a_{1+k}k\\ &-x^{1+k}a_{1+k}k^2-x^{k+2}a_{k+2}k^2+x^{-1+k}a_{-1+k}k-5x^{k+2}a_{k+2}k\\ &+3x^{k-2}a_{k-2}k+a_{k-2}x^{k-4}k^2-5a_{k-2}x^{k-4}k-3a_{-1+k}x^{-3+k}k\\ &+a_{-1+k}x^{-3+k}k^2+a_{1+k}x^{-1+k}k+a_{1+k}x^{-1+k}k^2+a_{k+2}x^kk^2\\ &+3a_{k+2}x^kk+l\,a_{k-2}x^{k-2}+l\,a_{-1+k}x^{-1+k}+l\,a_kx^k+l\,a_{1+k}x^{1+k}\\ &+l\,a_{k+2}x^{k+2}+l^2a_{k-2}x^{k-2}+l^2a_{-1+k}x^{-1+k}+l^2a_kx^k+l^2a_{1+k}x^{1+k}\\ &+l^2a_{k+2}x^{k+2}-a_kx^kk+6a_{k-2}x^{k-4}+2a_{-1+k}x^{-3+k}+2a_{k+2}x^k=0 \end{aligned} \quad (1.2)$$

```
> eq3:=map(coeff, eq2, x^k);
```

$$eq3 := 2a_{k+2}-a_kk^2+l\,a_k+3a_{k+2}k-a_kk+l^2a_k+a_{k+2}k^2=0 \quad (1.3)$$

A relação de recorrência é dada por

```
> eq4:=isolate(eq3,a[k+2]);
```

$$eq4 := a_{k+2}=\frac{a_kk^2-l\,a_k+a_kk-l^2a_k}{2+3k+k^2} \quad (1.4)$$

Quando l é inteiro uma das séries sempre termina. Calculemos explicitamente as séries neste caso.

```
> restart;
> y := sum(a[i]*x^i, i = 0 .. 8);
```

(1.5)

$$y := a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \quad \textbf{(1.5)}$$

```
> eq1:=(1-x^2)*(diff(y, x, x))-2*x*(diff(y,x))+l*(l+1)*y = 0;
```

$$eq1 := (1 - x^2)(2 a_2 + 6 a_3 x + 12 a_4 x^2 + 20 a_5 x^3 + 30 a_6 x^4 + 42 a_7 x^5 + 56 a_8 x^6) - 2x(a_1 + 2 a_2 x + 3 a_3 x^2 + 4 a_4 x^3 + 5 a_5 x^4 + 6 a_6 x^5 + 7 a_7 x^6 + 8 a_8 x^7) + l(l+1)(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8) = 0 \quad \textbf{(1.6)}$$

```
> eq2 := collect(eq1, x);
```

$$eq2 := (-72 a_8 + l(l+1) a_8) x^8 + (-56 a_7 + l(l+1) a_7) x^7 + (56 a_8 - 42 a_6 + l(l+1) a_6) x^6 + (-30 a_5 + 42 a_7 + l(l+1) a_5) x^5 + (l(l+1) a_4 - 20 a_4 + 30 a_6) x^4 + (-12 a_3 + 20 a_5 + l(l+1) a_3) x^3 + (12 a_4 - 6 a_2 + l(l+1) a_2) x^2 + (6 a_3 + l(l+1) a_1 - 2 a_1) x + 2 a_2 + l(l+1) a_0 = 0 \quad \textbf{(1.7)}$$

```
> for i from 0 to 6 do
>    g:=map(coeff,eq2,x,i);
     isolate(g,a[i+2]);
     assign(%);
  od;
```

$$g := 2 a_2 + l(l+1) a_0 = 0$$

$$a_2 = -\frac{1}{2} l(l+1) a_0$$

$$g := 6 a_3 + l(l+1) a_1 - 2 a_1 = 0$$

$$a_3 = -\frac{1}{6} l(l+1) a_1 + \frac{1}{3} a_1$$

$$g := 12 a_4 + 3 l(l+1) a_0 - \frac{1}{2} l^2 (l+1)^2 a_0 = 0$$

$$a_4 = -\frac{1}{4} l(l+1) a_0 + \frac{1}{24} l^2 (l+1)^2 a_0$$

$$g := 2 l(l+1) a_1 - 4 a_1 + 20 a_5 + l(l+1)\left(-\frac{1}{6} l(l+1) a_1 + \frac{1}{3} a_1\right) = 0$$

$$a_5 = -\frac{1}{10} l(l+1) a_1 + \frac{1}{5} a_1 - \frac{1}{20} l(l+1)\left(-\frac{1}{6} l(l+1) a_1 + \frac{1}{3} a_1\right)$$

$$g := l(l+1)\left(-\frac{1}{4} l(l+1) a_0 + \frac{1}{24} l^2 (l+1)^2 a_0\right) + 5 l(l+1) a_0 - \frac{5}{6} l^2 (l+1)^2 a_0 + 30 a_6 = 0$$

$$a_6 = -\frac{1}{30} l(l+1)\left(-\frac{1}{4} l(l+1) a_0 + \frac{1}{24} l^2 (l+1)^2 a_0\right) - \frac{1}{6} l(l+1) a_0 + \frac{1}{36} l^2 (l+1)^2 a_0$$

$$g := 3\,l\,(l+1)\,a_1 - 6\,a_1 + \frac{3}{2}\,l\,(l+1)\left(-\frac{1}{6}\,l\,(l+1)\,a_1 + \frac{1}{3}\,a_1\right) + 42\,a_7 + l\,(l+1)\left(-\frac{1}{10}\,l\,(l+1)\,a_1 + \frac{1}{5}\,a_1 - \frac{1}{20}\,l\,(l+1)\left(-\frac{1}{6}\,l\,(l+1)\,a_1 + \frac{1}{3}\,a_1\right)\right) = 0$$

$$a_7 = -\frac{1}{14}\,l\,(l+1)\,a_1 + \frac{1}{7}\,a_1 - \frac{1}{28}\,l\,(l+1)\left(-\frac{1}{6}\,l\,(l+1)\,a_1 + \frac{1}{3}\,a_1\right) - \frac{1}{42}\,l\,(l+1)\left(-\frac{1}{10}\,l\,(l+1)\,a_1 + \frac{1}{5}\,a_1 - \frac{1}{20}\,l\,(l+1)\left(-\frac{1}{6}\,l\,(l+1)\,a_1 + \frac{1}{3}\,a_1\right)\right)$$

$$g := 56\,a_8 + \frac{7}{5}\,l\,(l+1)\left(-\frac{1}{4}\,l\,(l+1)\,a_0 + \frac{1}{24}\,l^2\,(l+1)^2\,a_0\right) + 7\,l\,(l+1)\,a_0 - \frac{7}{6}\,l^2\,(l+1)^2\,a_0 + l\,(l+1)\left(-\frac{1}{30}\,l\,(l+1)\left(-\frac{1}{4}\,l\,(l+1)\,a_0 + \frac{1}{24}\,l^2\,(l+1)^2\,a_0\right) - \frac{1}{6}\,l\,(l+1)\,a_0 + \frac{1}{36}\,l^2\,(l+1)^2\,a_0\right) = 0$$

$$a_8 = -\frac{1}{40}\,l\,(l+1)\left(-\frac{1}{4}\,l\,(l+1)\,a_0 + \frac{1}{24}\,l^2\,(l+1)^2\,a_0\right) - \frac{1}{8}\,l\,(l+1)\,a_0 + \frac{1}{48}\,l^2\,(l+1)^2\,a_0 - \frac{1}{56}\,l\,(l+1)\left(-\frac{1}{30}\,l\,(l+1)\left(-\frac{1}{4}\,l\,(l+1)\,a_0 + \frac{1}{24}\,l^2\,(l+1)^2\,a_0\right) - \frac{1}{6}\,l\,(l+1)\,a_0 + \frac{1}{36}\,l^2\,(l+1)^2\,a_0\right) \tag{1.8}$$

Para l inteiro as série termina. Para cada valor de l temos um polinômio, denominado polinômio de Legendre. Para l par temos

```
> a[1] := 0: a[0] := 1:
> for l from 0 by 2 to 8 do P[l]:=y end do;
```

$$P_0 := 1$$

$$P_2 := 1 - 3\,x^2$$

$$P_4 := 1 - 10\,x^2 + \frac{35}{3}\,x^4$$

$$P_6 := 1 - 21\,x^2 + 63\,x^4 - \frac{231}{5}\,x^6$$

$$P_8 := 1 - 36\,x^2 + 198\,x^4 - \frac{1716}{5}\,x^6 + \frac{1287}{7}\,x^8 \tag{1.9}$$

```
> a[1] := 1; a[0] := 0;
```

$$a_1 := 1$$

$$a_0 := 0 \tag{1.10}$$

Para l ímpar,

```
> for l from 1 by 2 to 7 do P[l] := y end do;
```

$$P_1 := x$$

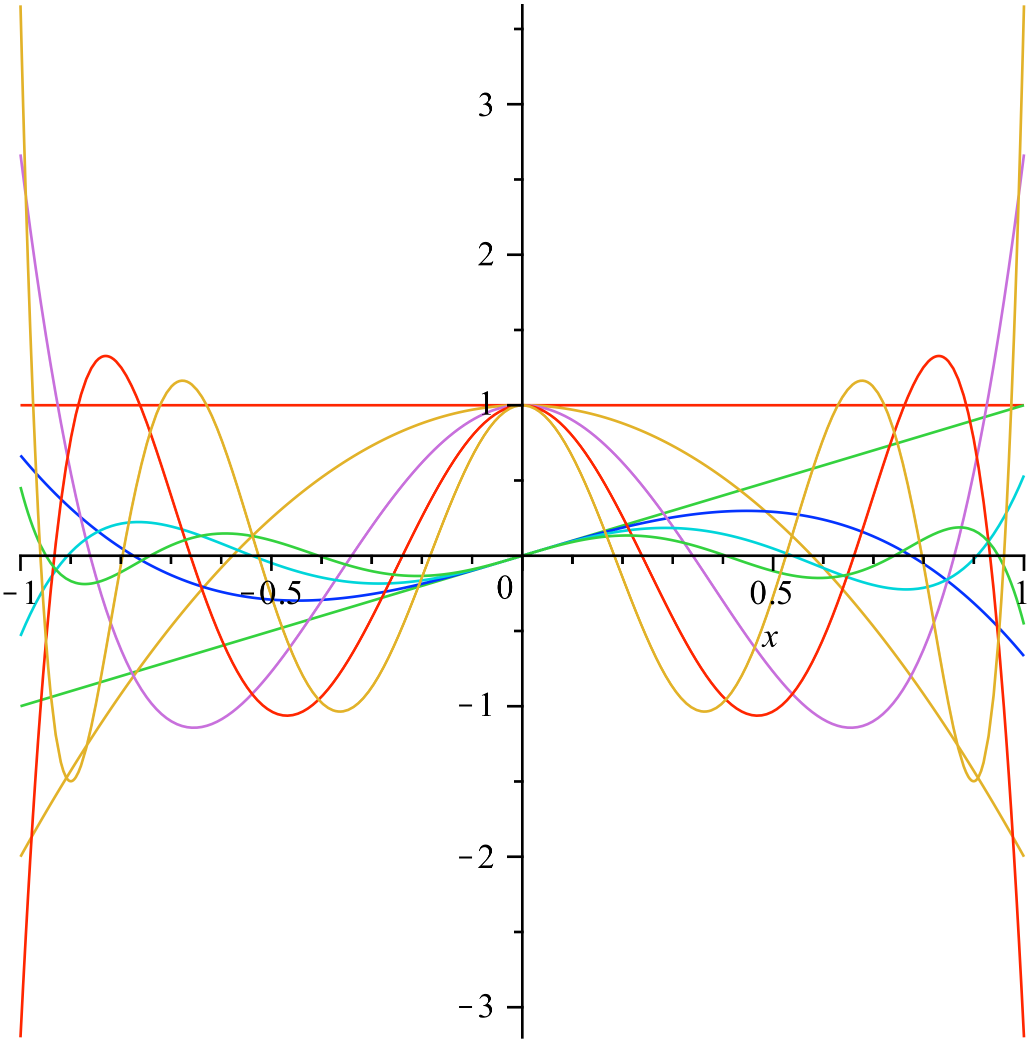
$$P_3 := x - \frac{5}{3}\,x^3$$

$$P_5 := x - \frac{14}{3} x^3 + \frac{21}{5} x^5$$

$$P_7 := x - 9 x^3 + \frac{99}{5} x^5 - \frac{429}{35} x^7 \qquad \textbf{(1.11)}$$

Façamos os respectivos gráficos:

```
> plot([seq(P[j],j=0..8)],x=-1..1);
```



No Maple os polinômios de Legendre podem ser gerados da seguinte forma:

```
> for j from 0 to 8 do
    p[j] := expand(LegendreP(j, x));
  od;
```

$$p_0 := 1$$

$$p_1 := x$$

$$p_2 := -\frac{1}{2} + \frac{3}{2} x^2$$

$$p_3 := \frac{5}{2} x^3 - \frac{3}{2} x$$

$$p_4 := \frac{3}{8} + \frac{35}{8}x^4 - \frac{15}{4}x^2$$

$$p_5 := \frac{63}{8}x^5 - \frac{35}{4}x^3 + \frac{15}{8}x$$

$$p_6 := -\frac{5}{16} + \frac{231}{16}x^6 - \frac{315}{16}x^4 + \frac{105}{16}x^2$$

$$p_7 := \frac{429}{16}x^7 - \frac{693}{16}x^5 + \frac{315}{16}x^3 - \frac{35}{16}x$$

$$p_8 := \frac{35}{128} + \frac{6435}{128}x^8 - \frac{3003}{32}x^6 + \frac{3465}{64}x^4 - \frac{315}{32}x^2 \quad \textbf{(1.12)}$$

Notemos que estes polinômios diferem daqueles calculados anteriormente somente por fatores constantes.

Vejamos como o Maple resolve a equação de Legendre com $l = 8$:

```
> x:='x':
> l := 8:
> Eq1 := (1-x^2)*(diff(Y(x), x, x))-2*x*(diff(Y(x), x))+l*(l+1)*Y
  (x) = 0;
```

$$Eq1 := (1 - x^2)\left(\frac{d^2}{dx^2}Y(x)\right) - 2x\left(\frac{d}{dx}Y(x)\right) + 72\,Y(x) = 0 \quad \textbf{(1.13)}$$

```
> dsolve(Eq1, Y(x));
```

$$Y(x) = \_C1\left(\frac{35}{128} + \frac{6435}{128}x^8 - \frac{3003}{32}x^6 + \frac{3465}{64}x^4 - \frac{315}{32}x^2\right) + \_C2\left(\frac{1}{8960}\left(1225 + 225225\,x^8 - 420420\,x^6 + 242550\,x^4 - 44100\,x^2\right)\ln\left(\frac{-x-1}{x-1}\right) - \frac{4213}{128}x^3 + \frac{15159}{4480}x - \frac{6435}{128}x^7 + \frac{9867}{128}x^5\right) \quad \textbf{(1.14)}$$

```
> yy := op(2, %);
```

$$yy := \_C1\left(\frac{35}{128} + \frac{6435}{128}x^8 - \frac{3003}{32}x^6 + \frac{3465}{64}x^4 - \frac{315}{32}x^2\right) + \_C2\left(\frac{1}{8960}\left(1225 + 225225\,x^8 - 420420\,x^6 + 242550\,x^4 - 44100\,x^2\right)\ln\left(\frac{-x-1}{x-1}\right) - \frac{4213}{128}x^3 + \frac{15159}{4480}x - \frac{6435}{128}x^7 + \frac{9867}{128}x^5\right) \quad \textbf{(1.15)}$$

Apliquemos condições iniciais e façamos o correspondente gráfico.

```
> EE1:=subs(x=0.8,yy)=1;
```

$$EE1 := -0.016655300\,\_C1 + \_C2\,(-0.008327651786\ln(9.000000000) + 0.57138029) = 1 \quad \textbf{(1.16)}$$

```
> EE2 := subs(x = 0.8, diff(yy, x)) = -2;
```

$$EE2 := -5.02948800\,\_C1 + \_C2\,(-2.514743996\ln(9.000000000) + 5.7622806) = -2 \quad \textbf{(1.17)}$$

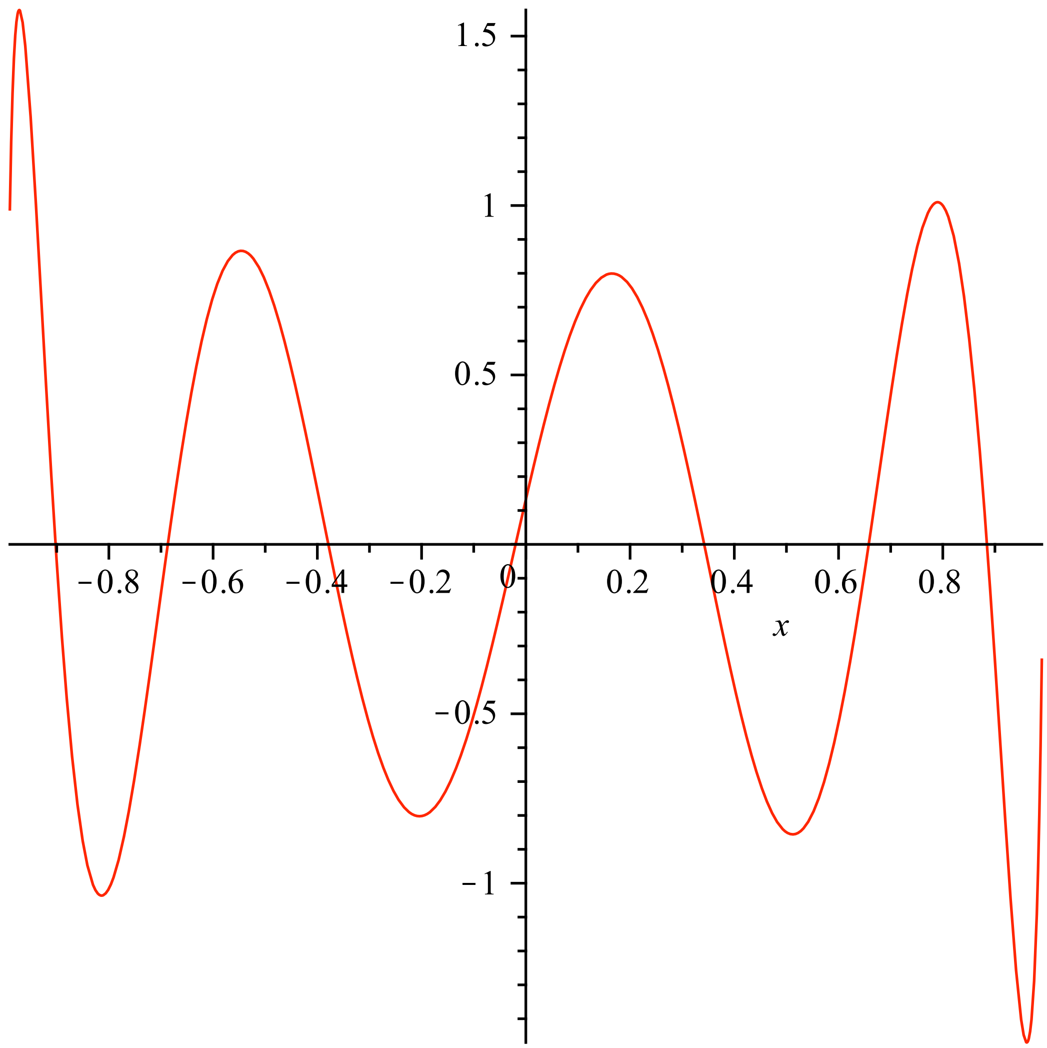
```
> sol := solve({EE1, EE2});
```

$$sol := \{\_C1 = 0.4834758325, \_C2 = 1.822607495\} \quad \textbf{(1.18)}$$

```
> assign(sol);
> yy
```

$$0.1322004229 + 24.30599205 x^8 - 45.37118516 x^6 + 26.17568374 x^4 - 4.759215226 x^2 + 0.0002034160151 \left(1225 + 225225 x^8 - 420420 x^6 + 242550 x^4 - 44100 x^2\right) \ln\left(\frac{-x-1}{x-1}\right) - 59.98941700 x^3 + 6.167166745 x - 91.62874399 x^7 + 140.4974074 x^5 \quad \textbf{(1.19)}$$

```
> plot(yy, x = -0.99 .. 0.99);
```



Notemos que esta solução pode ser gerada através de "força bruta", usando a relação de recorrência obtida anteriormente:

```
> l;
```

$$8 \quad \textbf{(1.20)}$$

$$> a_{k+2} = \frac{a_k k^2 - l a_k + a_k k - l^2 a_k}{2 + 3k + k^2}$$

$$a_{k+2} = \frac{a_k k - 72 a_k + a_k k^2}{2 + 3k + k^2} \quad \textbf{(1.21)}$$

```
> k := 'k': a_1 := 'a_1': a_0 := 'a_0':
> for k to 20 do
```

$$a_{k+2} := \frac{a_k k^2 - l\, a_k + a_k k - l^2 a_k}{2 + 3k + k^2}$$

od;

$$a_3 := -\frac{35}{3} a_1$$

$$a_4 := 198\, a_0$$

$$a_5 := 35\, a_1$$

$$a_6 := -\frac{1716}{5} a_0$$

$$a_7 := -35\, a_1$$

$$a_8 := \frac{1287}{7} a_0$$

$$a_9 := \frac{70}{9} a_1$$

$$a_{10} := 0$$

$$a_{11} := \frac{14}{11} a_1$$

$$a_{12} := 0$$

$$a_{13} := \frac{70}{143} a_1$$

$$a_{14} := 0$$

$$a_{15} := \frac{10}{39} a_1$$

$$a_{16} := 0$$

$$a_{17} := \frac{35}{221} a_1$$

$$a_{18} := 0$$

$$a_{19} := \frac{35}{323} a_1$$

$$a_{20} := 0$$

$$a_{21} := \frac{77}{969} a_1$$

$$a_{22} := 0 \qquad \textbf{(1.22)}$$

Vamos gerar coeficientes da série até $O(20)$:

A série é então dada por

```
> sum(a[m]*x^m,m=0..20);
```

$$a_0 + a_1 x - 36 a_0 x^2 - \frac{35}{3} a_1 x^3 + 198 a_0 x^4 + 35 a_1 x^5 - \frac{1716}{5} a_0 x^6 - 35 a_1 x^7 + \frac{1287}{7} a_0 x^8 + \frac{70}{9} a_1 x^9 + \frac{14}{11} a_1 x^{11} + \frac{70}{143} a_1 x^{13} + \frac{10}{39} a_1 x^{15} + \frac{35}{221} a_1 x^{17} + \frac{35}{323} a_1 x^{19} \quad \textbf{(1.23)}$$

```
> collect(%,a[1]):Y:=collect(%,a[0]);
```

$$Y := \left(1 - 36 x^2 + 198 x^4 - \frac{1716}{5} x^6 + \frac{1287}{7} x^8\right) a_0 + \left(x - \frac{35}{3} x^3 + 35 x^5 - 35 x^7 + \frac{70}{9} x^9 + \frac{14}{11} x^{11} + \frac{70}{143} x^{13} + \frac{10}{39} x^{15} + \frac{35}{221} x^{17} + \frac{35}{323} x^{19}\right) a_1 \quad \textbf{(1.24)}$$

Apliquermos agora as mesmas condições inciais e façamos o gráfico, comparando com a solução dada pelo Maple:

```
> ee1:=subs(x=0.8,Y)=1;
```

$$ee1 := -0.06091081\, a_0 + 0.1497373445\, a_1 = 1 \quad \textbf{(1.25)}$$

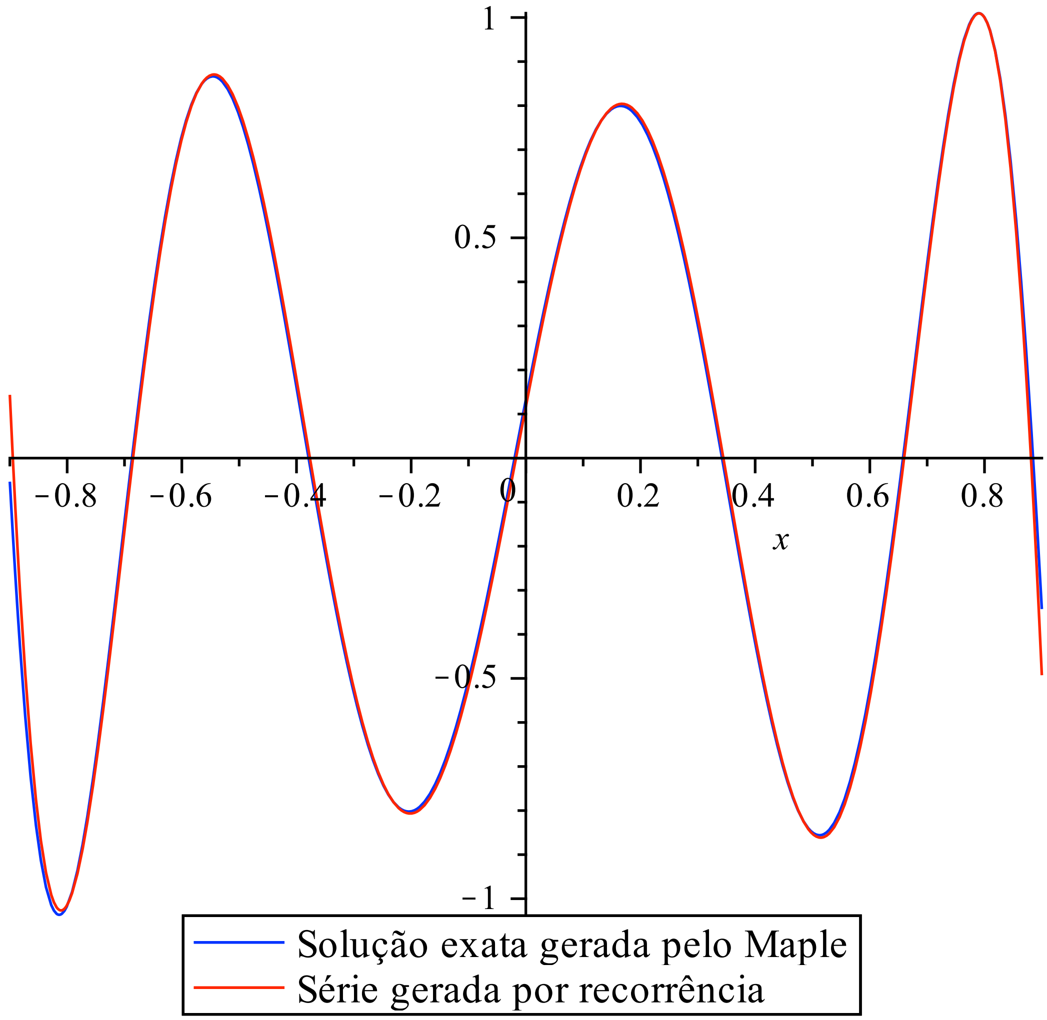
```
> ee2 := subs(x = 0.8, diff(Y, x)) = -2;
```

$$ee2 := -18.3935561\, a_0 + 0.02134134212\, a_1 = -2 \quad \textbf{(1.26)}$$

```
> sol := solve({ee1, ee2});
```

$$sol := \{a_0 = 0.1165373823, a_1 = 6.725766306\} \quad \textbf{(1.27)}$$

```
> assign(sol);
> plot([yy, Y], x = -0.9 .. 0.9, color = [blue, red], legend =
  ["Solução exata gerada pelo Maple", "Série gerada por
  recorrência"]);
```



Método de Frobenius

Consideremos a seguinte EDO linear:

$$P(x)\,\frac{d^2}{dx^2}\,y(x) + Q(x)\,\frac{d}{dx}\,y(x) + R(x)\,y(x) = 0\,.$$

Suponhamos que $x = 0$ é um ponto singular regular desta equação, ou seja, $P(0) = 0$, e $x\,\frac{Q(x)}{P(x)}$, $x^2\,\frac{R(x)}{P(x)}$ são analíticos em $x = 0$.

Buscaremos uma solução desta equação da forma

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r},$$

de modo que

$$\frac{d}{dx} y(x) = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1},$$

$$\frac{d^2}{dx^2} y(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1)\, c_n x^{n+r-2}.$$

Substituindo estas séries na EDO temos

$$P(x)\left(\sum_{n=0}^{\infty} (n+r)(n+r-1)\, c_n x^{n+r-2}\right) + Q(x)\left(\sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

.

Definimos

$$S_2 = P(x)\left(\sum_{n=0}^{\infty} (n+r)(n+r-1)\, c_n x^{n+r-2}\right),$$

$$S_1 = Q(x)\left(\sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}\right),$$

$$S_0 = \sum_{n=0}^{\infty} c_n x^{n+r},$$

de modo que a EDO torna-se $S_2 + S_1 + S_0 = 0$. Consideremos alguns exemplos específicos.

Exemplo 1

Busquemos a solução da equação

$$3\,x\, \frac{d^2}{dx^2} y(x) + \frac{d}{dx} y(x) - y(x) = 0\,.$$

em torno do ponto singular regular $x = 0$.

```
> restart;
> P := 3*x: Q:=1: R:=-1:
> S2 := sum(simplify(P*c[n]*(n+r)*(n+r-1)*x^(n+r-2)), n = 0 .. k)
;
```

$$S2 := \sum_{n=0}^{k} 3\, x^{n+r-1}\, c_n\, (n+r)\, (n+r-1) \tag{3.1}$$

```
> S1 := sum(simplify(Q*c[n]*(n+r)*x^(n+r-1)), n = 0 .. k);
```

$$S1 := \sum_{n=0}^{k} x^{n+r-1}\, c_n\, (n+r) \tag{3.2}$$

```
> S0 := sum(simplify(R*c[n]*x^(n+r)), n = 0 .. k);
```

$$S0 := \sum_{n=0}^{k} \left(-c_n\, x^{n+r}\right) \tag{3.3}$$

A equação pode agora ser escrita como

```
> S := S2+S1+S0;
```

$$S := \sum_{n=0}^{k} 3\, x^{n+r-1}\, c_n\, (n+r)\, (n+r-1) + \sum_{n=0}^{k} x^{n+r-1}\, c_n\, (n+r) + \sum_{n=0}^{k} \left(-c_n\, x^{n+r}\right) \tag{3.4}$$

```
> k := 6;
```

$$k := 6 \tag{3.5}$$

```
> eq1 := simplify(S/x^r);
```

$$eq1 := \frac{1}{x}\Big(96\, c_6\, x^6 + c_1\, x + 21\, c_3\, x^3 + 8\, c_2\, x^2 + 40\, c_4\, x^4 + 65\, c_5\, x^5 - 2\, c_0\, r + 3\, c_0\, r^2 + 10\, x^2\, c_2\, r + 3\, x^2\, c_2\, r^2 + 16\, x^3\, c_3\, r + 3\, x^3\, c_3\, r^2 + 22\, x^4\, c_4\, r + 3\, x^4\, c_4\, r^2 + 28\, x^5\, c_5\, r + 3\, x^5\, c_5\, r^2 + 34\, x^6\, c_6\, r + 3\, x^6\, c_6\, r^2 + 3\, c_1\, r^2\, x + 4\, c_1\, r\, x - c_0\, x - c_6\, x^7 - c_1\, x^2 - c_3\, x^4 - c_2\, x^3 - c_4\, x^5 - c_5\, x^6\Big) \tag{3.6}$$

```
> eq2 := x*%:
> eq3 := collect(%, x);
```

$$eq3 := -c_6\, x^7 + \left(96\, c_6 + 3\, c_6\, r^2 + 34\, c_6\, r - c_5\right) x^6 + \left(-c_4 + 28\, c_5\, r + 65\, c_5 + 3\, c_5\, r^2\right) x^5 + \left(3\, c_4\, r^2 + 40\, c_4 - c_3 + 22\, c_4\, r\right) x^4 + \left(-c_2 + 21\, c_3 + 16\, c_3\, r + 3\, c_3\, r^2\right) x^3 + \left(10\, c_2\, r - c_1 + 3\, c_2\, r^2 + 8\, c_2\right) x^2 + \left(-c_0 + c_1 + 3\, c_1\, r^2 + 4\, c_1\, r\right) x - 2\, c_0\, r + 3\, c_0\, r^2 \tag{3.7}$$

```
> e0 := coeff(eq3, x, 0);
```

$$e0 := -2\, c_0\, r + 3\, c_0\, r^2 \tag{3.8}$$

```
> s := solve(e0, r);
```

$$\tag{3.9}$$

$$s := 0, \frac{2}{3} \quad \textbf{(3.9)}$$

```
> r1 := s[1]; r2 := s[2];
```

$$r1 := 0$$

$$r2 := \frac{2}{3} \quad \textbf{(3.10)}$$

Como $r2 - r1 = 2/3$, as relações de recorrências associadas a cada raiz geram soluções linearmente independentes. Determinemos as relações de recorrência para r indeterminado:

```
> eqs := array(1 .. 6);
  for n from 1 to 6 do
      eqs[n] := coeff(eq3, x, n) = 0
  end do;
  coeffEqs := convert(eqs, list);
```

$$eqs := array(1..6, [\ ])$$

$$eqs_1 := -c_0 + c_1 + 3\, c_1\, r^2 + 4\, c_1\, r = 0$$

$$eqs_2 := 10\, c_2\, r - c_1 + 3\, c_2\, r^2 + 8\, c_2 = 0$$

$$eqs_3 := -c_2 + 21\, c_3 + 16\, c_3\, r + 3\, c_3\, r^2 = 0$$

$$eqs_4 := 3\, c_4\, r^2 + 40\, c_4 - c_3 + 22\, c_4\, r = 0$$

$$eqs_5 := -c_4 + 28\, c_5\, r + 65\, c_5 + 3\, c_5\, r^2 = 0$$

$$eqs_6 := 96\, c_6 + 3\, c_6\, r^2 + 34\, c_6\, r - c_5 = 0$$

$$coeffEqs := \left[-c_0 + c_1 + 3\, c_1\, r^2 + 4\, c_1\, r = 0, 10\, c_2\, r - c_1 + 3\, c_2\, r^2 + 8\, c_2 = 0, -c_2 + 21\, c_3 + 16\, c_3\, r + 3\, c_3\, r^2 = 0, 3\, c_4\, r^2 + 40\, c_4 - c_3 + 22\, c_4\, r = 0, -c_4 + 28\, c_5\, r + 65\, c_5 + 3\, c_5\, r^2 = 0, 96\, c_6 + 3\, c_6\, r^2 + 34\, c_6\, r - c_5 = 0 \right] \quad \textbf{(3.11)}$$

Temos aqui 6 equações para 6 incógnitas c_1 a c_6.

```
> s1 := [seq(c[n], n = 1 .. 6)];
```

$$s1 := [c_1, c_2, c_3, c_4, c_5, c_6] \quad \textbf{(3.12)}$$

```
>
```

Resolvendo estas equações para a_0 obtemos

```
> coef := solve(coeffEqs, s1);
```

$$coef := \left[\left[c_1 = \frac{c_0}{1 + 3\, r^2 + 4\, r}, c_2 = \frac{c_0}{42\, r + 67\, r^2 + 8 + 42\, r^3 + 9\, r^4}, c_3 = \frac{c_0}{1010\, r + 2103\, r^2 + 2080\, r^3 + 168 + 270\, r^5 + 27\, r^6 + 1062\, r^4}, c_4 = c_0 \Big/ \right.\right. \quad \textbf{(3.13)}$$

$(81 r^8 + 1404 r^7 + 10206 r^6 + 40404 r^5 + 94549 r^4 + 132496 r^3 + 106844 r^2$

$+ 44096 r + 6720), c_5 = c_0 / (243 r^{10} + 6480 r^9 + 75195 r^8 + 498240 r^7$

$+ 2078349 r^6 + 5671120 r^5 + 10176105 r^4 + 11736160 r^3 + 8199708 r^2$

$+ 3054400 r + 436800), c_6 = c_0 / (41932800 + 892331968 r^2 + 308073600 r$

$+ 925623570 r^5 + 1400534644 r^4 + 422867899 r^6 + 135508266 r^7 + 30393927 r^8$

$+ 4673430 r^9 + 469233 r^{10} + 27702 r^{11} + 729 r^{12} + 1414624632 r^3)]]$

> *assign*(*coef*)

```
> y := x^r*(sum(c[i]*x^i, i = 0 .. 6));
```

$$y := x^r \left(c_0 + \frac{c_0 x}{1 + 3 r^2 + 4 r} + \frac{c_0 x^2}{42 r + 67 r^2 + 8 + 42 r^3 + 9 r^4} + \frac{c_0 x^3}{1010 r + 2103 r^2 + 2080 r^3 + 168 + 270 r^5 + 27 r^6 + 1062 r^4} + (c_0 x^4) / (81 r^8 + 1404 r^7 + 10206 r^6 + 40404 r^5 + 94549 r^4 + 132496 r^3 + 106844 r^2 + 44096 r + 6720) + (c_0 x^5) / (243 r^{10} + 6480 r^9 + 75195 r^8 + 498240 r^7 + 2078349 r^6 + 5671120 r^5 + 10176105 r^4 + 11736160 r^3 + 8199708 r^2 + 3054400 r + 436800) + (c_0 x^6) / (41932800 + 892331968 r^2 + 308073600 r + 925623570 r^5 + 1400534644 r^4 + 422867899 r^6 + 135508266 r^7 + 30393927 r^8 + 4673430 r^9 + 469233 r^{10} + 27702 r^{11} + 729 r^{12} + 1414624632 r^3)\right) \quad \textbf{(3.14)}$$

> $y1 := simplify\left(\frac{subs(r = r1, y)}{c[0]}\right)$

$$y1 := 1 + x + \frac{1}{8} x^2 + \frac{1}{168} x^3 + \frac{1}{6720} x^4 + \frac{1}{436800} x^5 + \frac{1}{41932800} x^6 \quad \textbf{(3.15)}$$

> $y2 := simplify\left(\frac{subs(r = r2, y)}{c[0]}\right)$

$$y2 := \frac{1}{1507968000} x^{2/3} (1507968000 + 301593600 x + 18849600 x^2 + 571200 x^3 + 10200 x^4 + 120 x^5 + x^6) \quad \textbf{(3.16)}$$

> $Y := C1 \cdot y1 + C2 \cdot y2$

$$Y := C1\left(1 + x + \frac{1}{8}x^2 + \frac{1}{168}x^3 + \frac{1}{6720}x^4 + \frac{1}{436800}x^5 + \frac{1}{41932800}x^6\right) + \frac{1}{1507968000}C2\, x^{2/3}\left(1507968000 + 301593600\, x + 18849600\, x^2 + 571200\, x^3 + 10200\, x^4 + 120\, x^5 + x^6\right) \quad \textbf{(3.17)}$$

Determinemos a solução correspondente às condições iniciais $y(1) = 1$ e $y'(1) = -2$.

```
> ee1:=subs(x=1,Y)=1;
```

$$ee1 := \frac{89363137}{41932800}C1 + \frac{1828992721}{1507968000}C2 = 1 \quad \textbf{(3.18)}$$

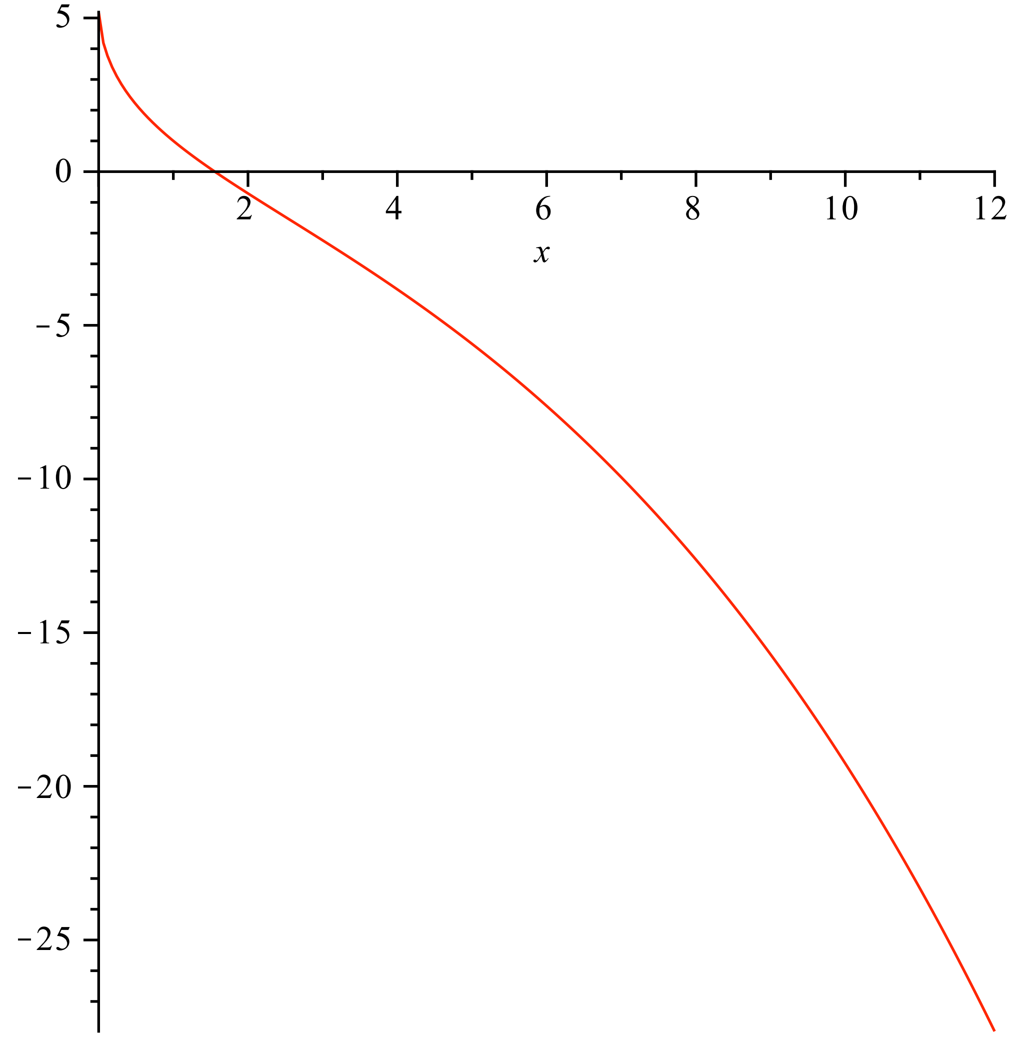
```
> ee2 := subs(x = 1, diff(Y, x)) = -2;
```

$$ee2 := \frac{8865041}{6988800}C1 + \frac{234056443}{226195200}C2 = -2 \quad \textbf{(3.19)}$$

```
> sol := solve({ee1, ee2});
```

$$sol := \left\{C1 = \frac{328230778043750400}{63233320672117861}, C2 = -\frac{524584036247040000}{63233320672117861}\right\} \quad \textbf{(3.20)}$$

```
> assign(sol);
> plot(Y, x = 0 .. 12);
```



Comparemos este resultado com aquele fornecido pelos comandos do Maple:

```
> eq := 3*x*(diff(yy(x), `$`(x, 2)))+diff(yy(x), x)-yy(x) = 0;
```

$$eq := 3\,x\left(\frac{d^2}{dx^2}\,yy(x)\right) + \frac{d}{dx}\,yy(x) - yy(x) = 0 \quad \textbf{(3.21)}$$

A solução geral é dada por

```
> dsolve(eq, yy(x));
```

$$yy(x) = \_C1\,x^{1/3}\,\mathrm{BesselI}\left(\frac{2}{3}, \frac{2}{3}\sqrt{3}\,\sqrt{x}\right) + \_C2\,x^{1/3}\,\mathrm{BesselK}\left(\frac{2}{3}, \frac{2}{3}\sqrt{3}\,\sqrt{x}\right) \quad \textbf{(3.22)}$$

Usando as condições inciais temos

```
> dsolve({eq, yy(1) = 1, (D(yy))(1) = -2}, yy(x));
```

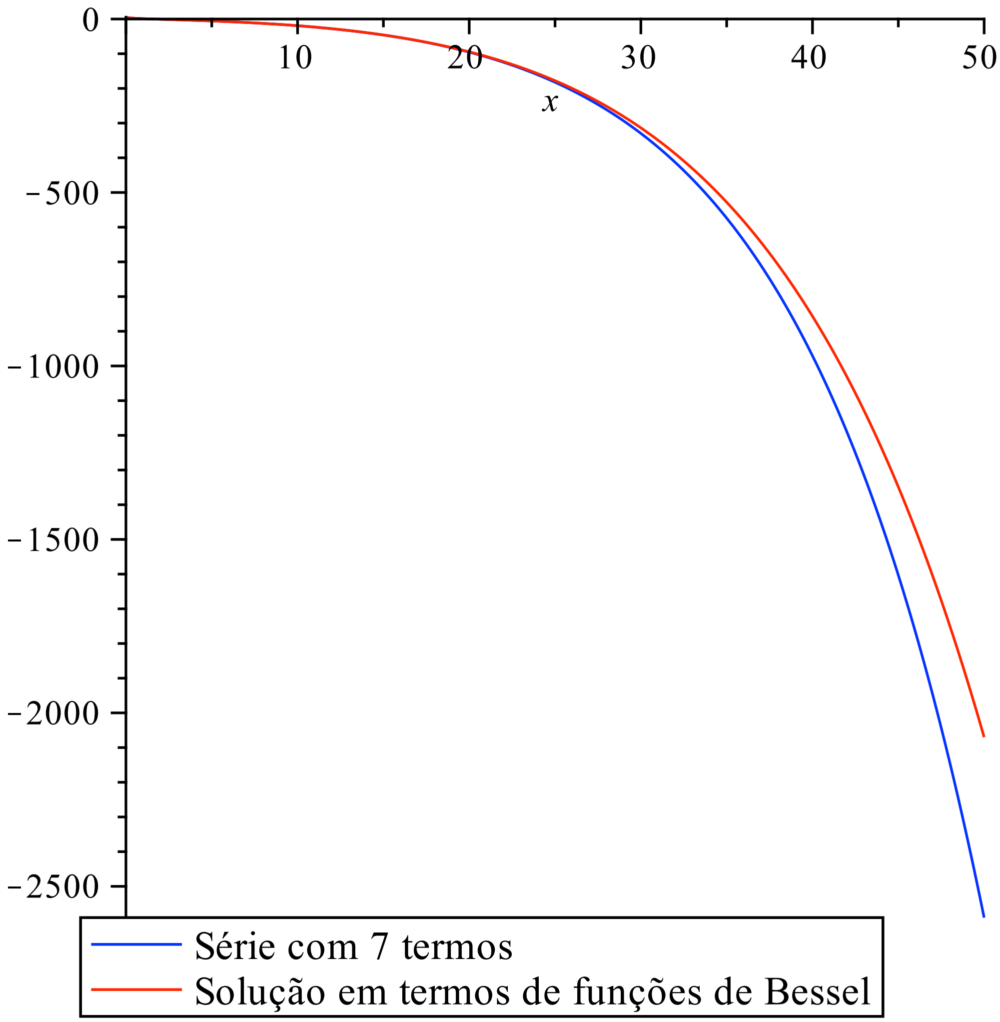
$$yy(x) = \frac{1}{9}\left(\left(-18\,\mathrm{BesselK}\left(\frac{2}{3}, \frac{2}{3}\sqrt{3}\right) + 3\sqrt{3}\,\mathrm{BesselK}\left(\frac{1}{3}, \frac{2}{3}\sqrt{3}\right)\right)\sqrt{3}\,x^{1/3}\,\mathrm{BesselI}\left(\frac{2}{3}, \frac{2}{3}\sqrt{3}\,\sqrt{x}\right)\right) \Big/ \left(\mathrm{BesselK}\left(\frac{2}{3}, \frac{2}{3}\sqrt{3}\right)\mathrm{BesselI}\left(-\frac{1}{3}, \frac{2}{3}\sqrt{3}\right) + \mathrm{BesselK}\left(\frac{1}{3}, \frac{2}{3}\sqrt{3}\right)\mathrm{BesselI}\left(\frac{2}{3}, \frac{2}{3}\sqrt{3}\right)\right) - \frac{1}{9}\left(\sqrt{3}\left(\right.\right. \quad \textbf{(3.23)}$$

$$-18\,\mathrm{BesselI}\left(\frac{2}{3}, \frac{2}{3}\sqrt{3}\right) - 3\sqrt{3}\,\mathrm{BesselI}\left(-\frac{1}{3}, \frac{2}{3}\sqrt{3}\right)\Bigg) x^{1/3}\,\mathrm{BesselK}\left(\frac{2}{3}, \frac{2}{3}\sqrt{3}\sqrt{x}\right)\Bigg) \Bigg/ \left(\mathrm{BesselK}\left(\frac{2}{3}, \frac{2}{3}\sqrt{3}\right)\mathrm{BesselI}\left(-\frac{1}{3}, \frac{2}{3}\sqrt{3}\right) + \mathrm{BesselK}\left(\frac{1}{3}, \frac{2}{3}\sqrt{3}\right)\mathrm{BesselI}\left(\frac{2}{3}, \frac{2}{3}\sqrt{3}\right)\right)\Bigg)$$

```
> YY := op(2, %);
```

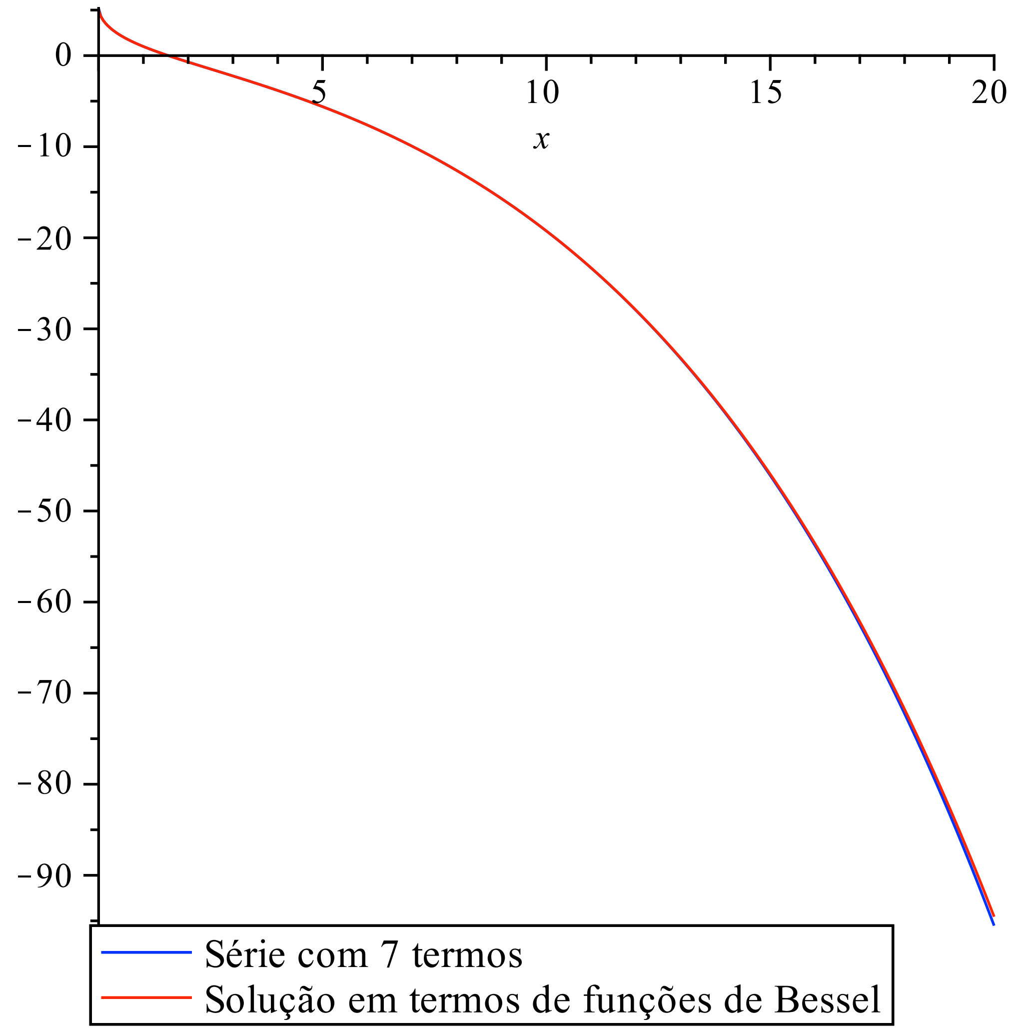
$$YY := \frac{1}{9}\left(\left(\left(-18\,\mathrm{BesselK}\left(\frac{2}{3}, \frac{2}{3}\sqrt{3}\right) + 3\sqrt{3}\,\mathrm{BesselK}\left(\frac{1}{3}, \frac{2}{3}\sqrt{3}\right)\right)\sqrt{3}\,x^{1/3}\,\mathrm{BesselI}\left(\frac{2}{3}, \frac{2}{3}\sqrt{3}\sqrt{x}\right)\right) \Bigg/ \left(\mathrm{BesselK}\left(\frac{2}{3}, \frac{2}{3}\sqrt{3}\right)\mathrm{BesselI}\left(-\frac{1}{3}, \frac{2}{3}\sqrt{3}\right) + \mathrm{BesselK}\left(\frac{1}{3}, \frac{2}{3}\sqrt{3}\right)\mathrm{BesselI}\left(\frac{2}{3}, \frac{2}{3}\sqrt{3}\right)\right)\right) - \frac{1}{9}\left(\sqrt{3}\left(-18\,\mathrm{BesselI}\left(\frac{2}{3}, \frac{2}{3}\sqrt{3}\right) - 3\sqrt{3}\,\mathrm{BesselI}\left(-\frac{1}{3}, \frac{2}{3}\sqrt{3}\right)\right) x^{1/3}\,\mathrm{BesselK}\left(\frac{2}{3}, \frac{2}{3}\sqrt{3}\sqrt{x}\right)\right) \Bigg/ \left(\mathrm{BesselK}\left(\frac{2}{3}, \frac{2}{3}\sqrt{3}\right)\mathrm{BesselI}\left(-\frac{1}{3}, \frac{2}{3}\sqrt{3}\right) + \mathrm{BesselK}\left(\frac{1}{3}, \frac{2}{3}\sqrt{3}\right)\mathrm{BesselI}\left(\frac{2}{3}, \frac{2}{3}\sqrt{3}\right)\right)\right) \tag{3.24}$$

```
> plot([Y, YY], x = 0 .. 50, color = [blue, red], legend =
  ["Série com 7 termos", "Solução em termos de funções de
  Bessel"]);
```



Notemos que a concordância é boa até $x \approx 20$.

```
> plot([Y, YY], x = 0 .. 20, color = [blue, red], legend =
  ["Série com 7 termos", "Solução em termos de funções de
  Bessel"]);
```



Exemplo 2

Busquemos a solução da equação

$$x \frac{d^2}{dx^2} y(x) + 5 \frac{d}{dx} y(x) + x\, y(x) = 0 .$$

em torno do ponto singular regular $x = 0$.

```
> restart;
> P := x: Q:=5: R:=x:
> S2 := sum(simplify(P*c[n]*(n+r)*(n+r-1)*x^(n+r-2)), n = 0 .. k)
;
```

$$S2 := \sum_{n=0}^{k} x^{n+r-1} c_n (n+r)(n+r-1) \tag{4.1}$$

```
> S1 := sum(simplify(Q*c[n]*(n+r)*x^(n+r-1)), n = 0 .. k);
```

$$S1 := \sum_{n=0}^{k} 5\, x^{n+r-1}\, c_n\, (n+r) \quad \textbf{(4.2)}$$

```
> S0 := sum(simplify(R*c[n]*x^(n+r)), n = 0 .. k);
```

$$S0 := \sum_{n=0}^{k} x^{n+1+r}\, c_n \quad \textbf{(4.3)}$$

A equação pode agora ser escrita como

```
> S := S2+S1+S0;
```

$$S := \sum_{n=0}^{k} x^{n+r-1}\, c_n\, (n+r)\, (n+r-1) + \sum_{n=0}^{k} 5\, x^{n+r-1}\, c_n\, (n+r) + \sum_{n=0}^{k} x^{n+1+r}\, c_n \quad \textbf{(4.4)}$$

```
> k := 6;
```

$$k := 6 \quad \textbf{(4.5)}$$

```
> eq1 := simplify(S/x^r);
```

$$eq1 := \frac{1}{x} \big(4\, c_0\, r + c_0\, r^2 + 8\, x^2\, c_2\, r + x^2\, c_2\, r^2 + 10\, x^3\, c_3\, r + x^3\, c_3\, r^2 + 12\, x^4\, c_4\, r + x^4\, c_4\, r^2 + 14\, x^5\, c_5\, r + x^5\, c_5\, r^2 + 16\, x^6\, c_6\, r + x^6\, c_6\, r^2 + 6\, c_1\, r\, x + c_1\, r^2\, x + 5\, c_1\, x + x^6\, c_4 + x^4\, c_2 + x^8\, c_6 + 12\, x^2\, c_2 + 21\, x^3\, c_3 + 32\, x^4\, c_4 + 45\, x^5\, c_5 + 60\, x^6\, c_6 + x^2\, c_0 + x^3\, c_1 + x^5\, c_3 + x^7\, c_5\big) \quad \textbf{(4.6)}$$

```
> eq2 := x*%:
> eq3 := collect(%, x);
```

$$eq3 := x^8\, c_6 + x^7\, c_5 + \left(c_4 + 60\, c_6 + 16\, c_6\, r + c_6\, r^2\right) x^6 + \left(c_5\, r^2 + 45\, c_5 + 14\, c_5\, r + c_3\right) x^5 + \left(c_4\, r^2 + c_2 + 12\, c_4\, r + 32\, c_4\right) x^4 + \left(c_1 + 10\, c_3\, r + 21\, c_3 + c_3\, r^2\right) x^3 + \left(c_2\, r^2 + 8\, c_2\, r + 12\, c_2 + c_0\right) x^2 + \left(5\, c_1 + 6\, c_1\, r + c_1\, r^2\right) x + 4\, c_0\, r + c_0\, r^2 \quad \textbf{(4.7)}$$

```
> e0 := coeff(eq3, x, 0);
```

$$e0 := 4\, c_0\, r + c_0\, r^2 \quad \textbf{(4.8)}$$

```
> s := solve(e0, r);
```

$$s := 0, -4 \quad \textbf{(4.9)}$$

```
> r1 := s[1]; r2 := s[2];
```

$$r1 := 0$$
$$r2 := -4 \quad \textbf{(4.10)}$$

```
>
```

Como $r1 - r2 = 4$, as relações de recorrências associadas a cada raiz não geram soluções linearmente independentes.
P

```
> r := r1:
> eqs := array(1 .. 6);
```

```
for n from 1 to 6 do
    eqs[n] := coeff(eq3, x, n) = 0
end do;
coeffEqs := convert(eqs, list);
```

$$eqs := array(1..6, [\ ])$$

$$eqs_1 := 5\, c_1 = 0$$

$$eqs_2 := 12\, c_2 + c_0 = 0$$

$$eqs_3 := c_1 + 21\, c_3 = 0$$

$$eqs_4 := c_2 + 32\, c_4 = 0$$

$$eqs_5 := 45\, c_5 + c_3 = 0$$

$$eqs_6 := c_4 + 60\, c_6 = 0$$

$$coeffEqs := [5\, c_1 = 0, 12\, c_2 + c_0 = 0, c_1 + 21\, c_3 = 0, c_2 + 32\, c_4 = 0, 45\, c_5 + c_3 = 0, c_4 + 60\, c_6 = 0] \quad \textbf{(4.11)}$$

Temos aqui 6 equações para 6 incógnitas c_1 a c_6.

```
> s1 := [seq(c[n], n = 1 .. 6)];
```

$$s1 := [c_1, c_2, c_3, c_4, c_5, c_6] \quad \textbf{(4.12)}$$

```
>
```

Resolvendo estas equações para a_0 obtemos

```
> coef := solve(coeffEqs, s1);
```

$$coef := \left[\left[c_1 = 0, c_2 = -\frac{1}{12}\, c_0, c_3 = 0, c_4 = \frac{1}{384}\, c_0, c_5 = 0, c_6 = -\frac{1}{23040}\, c_0\right]\right] \quad \textbf{(4.13)}$$

```
> assign(coef)
> y1 := expand(x^r*(sum(c[i]*x^i, i = 0 .. 6))/c[0]);
```

$$y1 := 1 - \frac{1}{12}\, x^2 + \frac{1}{384}\, x^4 - \frac{1}{23040}\, x^6 \quad \textbf{(4.14)}$$

Encontremos agora a segunda solução. Como $r1 - r2 = 4 > 0$, devemos buscar uma solução da forma:

$$y2 = k\, y1 \ln x + x^{r2} \sum_{k=0}^{\infty} b_k x^k$$

```
> y2 := B*y1*ln(x)+x^r2*(sum(b[j]*x^j, j = 0 .. 6));
```

$$y2 := B\left(1 - \frac{1}{12}\, x^2 + \frac{1}{384}\, x^4 - \frac{1}{23040}\, x^6\right) \ln(x) + \frac{b_0 + b_1\, x + b_2\, x^2 + b_3\, x^3 + b_4\, x^4 + b_5\, x^5 + b_6\, x^6}{x^4} \quad \textbf{(4.15)}$$

```
>
```

Se $B = 0$

```
> r := r2
```

$$r := -4 \quad \textbf{(4.16)}$$

> coeff(eq3, x, 2)

$$\frac{4}{3} c_0 \tag{4.17}$$

O que implicaria $c_0 = 0$, de modo que devemos supor que $B \neq 0$.

```
> eq4:=x*(diff(y2, x, x))+5*(diff(y2, x))+x*y2;
```

$$eq4 := x \left(B \left(-\frac{1}{6} + \frac{1}{32} x^2 - \frac{1}{768} x^4 \right) \ln(x) + \frac{2 B \left(-\frac{1}{6} x + \frac{1}{96} x^3 - \frac{1}{3840} x^5 \right)}{x} - \frac{B \left(1 - \frac{1}{12} x^2 + \frac{1}{384} x^4 - \frac{1}{23040} x^6 \right)}{x^2} + \frac{20 \left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \right)}{x^6} - \frac{8 \left(b_1 + 2 b_2 x + 3 b_3 x^2 + 4 b_4 x^3 + 5 b_5 x^4 + 6 b_6 x^5 \right)}{x^5} + \frac{2 b_2 + 6 b_3 x + 12 b_4 x^2 + 20 b_5 x^3 + 30 b_6 x^4}{x^4} \right) + 5 B \left(-\frac{1}{6} x + \frac{1}{96} x^3 - \frac{1}{3840} x^5 \right) \ln(x) + \frac{5 B \left(1 - \frac{1}{12} x^2 + \frac{1}{384} x^4 - \frac{1}{23040} x^6 \right)}{x} - \frac{20 \left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \right)}{x^5} + \frac{5 \left(b_1 + 2 b_2 x + 3 b_3 x^2 + 4 b_4 x^3 + 5 b_5 x^4 + 6 b_6 x^5 \right)}{x^4} + x \left(B \left(1 - \frac{1}{12} x^2 + \frac{1}{384} x^4 - \frac{1}{23040} x^6 \right) \ln(x) + \frac{b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6}{x^4} \right) \tag{4.18}$$

> eq5 := expand(eq4)

$$eq5 := \frac{4 B}{x} + b_3 + 5 b_5 - \frac{1}{23040} B \ln(x) x^7 - \frac{2}{3} x B + 12 x b_6 + \frac{1}{32} B x^3 - \frac{1}{1440} B x^5 - \frac{3 b_1}{x^4} - \frac{4 b_2}{x^3} - \frac{3 b_3}{x^2} + \frac{b_0}{x^3} + \frac{b_1}{x^2} + \frac{b_2}{x} + x b_4 + x^2 b_5 + x^3 b_6 \tag{4.19}$$

O termo envolvendo logaritmo deve ser desprezado, pois envolve uma potência além de 6. Devemos agora igualar a zero os coeficientes de x :

$$> eq6 := expand\left(x^4 \cdot \left(eq5 + \frac{1}{23040} B \ln(x) x^7\right)\right)$$

$$eq6 := 4 B x^3 + x^4 b_3 + 5 b_5 x^4 - \frac{2}{3} B x^5 + 12 b_6 x^5 + \frac{1}{32} x^7 B - \frac{1}{1440} x^9 B - 3 b_1 - 4 b_2 x - 3 b_3 x^2 + x b_0 + x^2 b_1 + x^3 b_2 + x^5 b_4 + x^6 b_5 + x^7 b_6 \quad \textbf{(4.20)}$$

$> b1 := 0 :$

```
> eqs2 := array(1 .. 7);
  for n from 1 to 7 do
      eqs2[n] := coeff(eq6, x, n) = 0
  end do;
  coeffEqs := convert(eqs2, list);
```

$$eqs2 := array(1..7, [\ ])$$

$$eqs2_1 := -4 b_2 + b_0 = 0$$

$$eqs2_2 := -3 b_3 + b_1 = 0$$

$$eqs2_3 := 4 B + b_2 = 0$$

$$eqs2_4 := b_3 + 5 b_5 = 0$$

$$eqs2_5 := -\frac{2}{3} B + 12 b_6 + b_4 = 0$$

$$eqs2_6 := b_5 = 0$$

$$eqs2_7 := \frac{1}{32} B + b_6 = 0$$

$$coeffEqs := \left[-4 b_2 + b_0 = 0, -3 b_3 + b_1 = 0, 4 B + b_2 = 0, b_3 + 5 b_5 = 0, -\frac{2}{3} B + 12 b_6 + b_4 = 0, b_5 = 0, \frac{1}{32} B + b_6 = 0\right] \quad \textbf{(4.21)}$$

$> s2 := [B, seq(b[n], n = 1..6)];$

$$s2 := [B, b_1, b_2, b_3, b_4, b_5, b_6] \quad \textbf{(4.22)}$$

$> sol2 := solve(coeffEqs, s2)$

$$sol2 := \left[\left[B = -\frac{1}{16} b_0, b_1 = 0, b_2 = \frac{1}{4} b_0, b_3 = 0, b_4 = -\frac{25}{384} b_0, b_5 = 0, b_6 = \frac{1}{512} b_0\right]\right] \quad \textbf{(4.23)}$$

$> assign(sol2)$

Como devemos obter somente uma solução particular, fazemos $b_0 = 1$, de modo que

$> b_0 := 1$

$$b_0 := 1 \quad \textbf{(4.24)}$$

$> y2 := expand(y2)$

$$y2 := -\frac{1}{16} \ln(x) + \frac{1}{192} \ln(x) x^2 - \frac{1}{6144} \ln(x) x^4 + \frac{1}{368640} \ln(x) x^6 + \frac{1}{x^4} \quad \textbf{(4.25)}$$

$$+\frac{1}{4x^2}-\frac{25}{384}+\frac{1}{512}x^2$$

A solução geral é então, até O(6),

```
> Y :=C[1]*y1+C[2]*y2;
```

$$Y:=C_1\left(1-\frac{1}{12}x^2+\frac{1}{384}x^4-\frac{1}{23040}x^6\right)+C_2\left(-\frac{1}{16}\ln(x)+\frac{1}{192}\ln(x)x^2-\frac{1}{6144}\ln(x)x^4+\frac{1}{368640}\ln(x)x^6+\frac{1}{x^4}+\frac{1}{4x^2}-\frac{25}{384}+\frac{1}{512}x^2\right) \quad \textbf{(4.26)}$$

Vejamos como o Maple resolve esta equação.

```
> ics := y3(2) = 1, (D(y3))(2) = -1:
> eq4 := x*(diff(y3(x), x, x))+5*(diff(y3(x), x))+x*y3(x)=0;
```

$$eq4:=x\left(\frac{d^2}{dx^2}y3(x)\right)+5\left(\frac{d}{dx}y3(x)\right)+x\,y3(x)=0 \quad \textbf{(4.27)}$$

```
> dsolve({eq4, ics})
```

$$y3(x)=-\frac{4\,(\mathrm{BesselY}(2,2)-\mathrm{BesselY}(1,2))\,\mathrm{BesselJ}(2,x)}{(-\mathrm{BesselY}(2,2)\,\mathrm{BesselJ}(1,2)+\mathrm{BesselY}(1,2)\,\mathrm{BesselJ}(2,2))\,x^2}+\frac{4\,(\mathrm{BesselJ}(2,2)-\mathrm{BesselJ}(1,2))\,\mathrm{BesselY}(2,x)}{(-\mathrm{BesselY}(2,2)\,\mathrm{BesselJ}(1,2)+\mathrm{BesselY}(1,2)\,\mathrm{BesselJ}(2,2))\,x^2} \quad \textbf{(4.28)}$$

```
> P[maple] := plot(op(2, %), x = 0.8 ..3) :
```

Façamos uma comparação com o resultado obtido anteriormente:

```
> e1 := subs(x = 2, Y) = 1;
```

$$e1:=\frac{127}{180}C_1+C_2\left(-\frac{127}{2880}\ln(2)+\frac{13}{192}\right)=1 \quad \textbf{(4.29)}$$

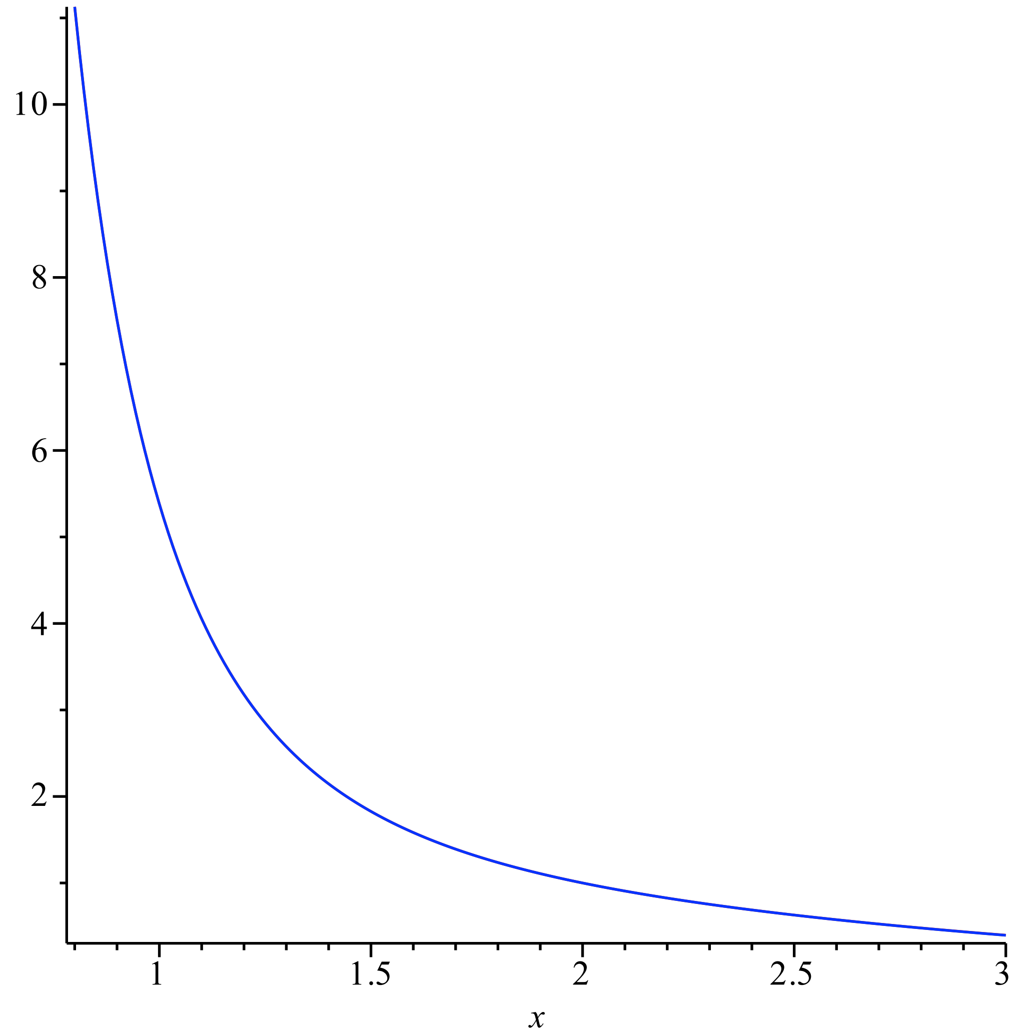
```
> e2 := subs(x = 2, diff(Y, x)) = -1;
```

$$e2:=-\frac{31}{120}C_1+C_2\left(-\frac{581}{2880}+\frac{31}{1920}\ln(2)\right)=-1 \quad \textbf{(4.30)}$$

```
> s3 := fsolve({e1, e2})
```

$$s3:=\{C_1=1.228744083,\ C_2=3.582227923\} \quad \textbf{(4.31)}$$

```
> assign(s3) :
> P[2] := plot(Y, x = 0.8 ..3, color = blue) :
> with(plots) :
> display([P[maple], P[2]])
```



Vemos que há excelente concordância neste intervalo.