Since g and ϕ are analytic at $z = z_0$ and $\phi(z_0) \neq 0$, it follows that the function g/ϕ is analytic at z_0. Moreover, $g(z_0) \neq 0$ implies $g(z_0)/\phi(z_0) \neq 0$. We conclude from Theorem 6.12 that the function f has a pole of order n at z_0. ✎

When $n = 1$ in (10), we see that a zero of order 1, or a simple zero, in the denominator h of $f(z) = g(z)/h(z)$ corresponds to a simple pole of f.

EXAMPLE 4 Order of Poles

(a) Inspection of the rational function

$$f(z) = \frac{2z+5}{(z-1)(z+5)(z-2)^4}$$

shows that the denominator has zeros of order 1 at $z = 1$ and $z = -5$, and a zero of order 4 at $z = 2$. Since the numerator is not zero at any of these points, it follows from Theorem 6.13 and (10) that f has simple poles at $z = 1$ and $z = -5$, and a pole of order 4 at $z = 2$.

(b) In Example 3 we saw that $z = 0$ is a zero of order 3 of $z \sin z^2$. From Theorem 6.13 and (10) we conclude that the reciprocal function $f(z) = 1/(z \sin z^2)$ has a pole of order 3 at $z = 0$.

Remarks

(i) From the preceding discussion, it should be intuitively clear that if a function f has a pole at $z = z_0$, then $|f(z)| \to \infty$ as $z \to z_0$ from any direction. From (i) of the Remarks following Section 2.6 we can write $\lim_{z \to z_0} f(z) = \infty$.

(ii) If you peruse other texts on complex variables, and you are encouraged to do this, you may encounter the term *meromorphic*. A function f is **meromorphic** if it is analytic throughout a domain D, except possibly for poles in D. It can be proved that a meromorphic function can have at most a finite number of poles in D. For example, the rational function $f(z) = 1/(z^2+1)$ is meromorphic in the complex plane.

EXERCISES 6.4

Answers to selected odd-numbered problems begin on page ANS-19.

In Problems 1–4, show that $z = 0$ is a removable singularity of the given function. Supply a definition of $f(0)$ so that f is analytic at $z = 0$.

1. $f(z) = \dfrac{e^{2z}-1}{z}$

2. $f(z) = \dfrac{z^3-4z^2}{1-e^{z^2/2}}$

3. $f(z) = \dfrac{\sin 4z - 4z}{z^2}$

4. $f(z) = \dfrac{1-\frac{1}{2}z^{10}-\cos z^5}{\sin z^2}$

In Problems 5–10, determine the zeros and their order for the given function.

5. $f(z) = (z + 2 - i)^2$

6. $f(z) = z^4 - 16$

7. $f(z) = z^4 + z^2$

8. $f(z) = \sin^2 z$

9. $f(z) = e^{2z} - e^z$

10. $f(z) = ze^z - z$

In Problems 11–14, the indicated number is a zero of the given function. Use a Maclaurin or Taylor series to determine the order of the zero.

11. $f(z) = z(1 - \cos^2 z);\ z = 0$

12. $f(z) = z - \sin z;\ z = 0$

13. $f(z) = 1 - e^{z-1};\ z = 1$

14. $f(z) = 1 - \pi i + z + e^z;\ z = \pi i$

In Problems 15–26, determine the order of the poles for the given function.

15. $f(z) = \dfrac{3z - 1}{z^2 + 2z + 5}$

16. $f(z) = 5 - \dfrac{6}{z^2}$

17. $f(z) = \dfrac{1 + 4i}{(z + 2)(z + i)^4}$

18. $f(z) = \dfrac{z - 1}{(z + 1)(z^3 + 1)}$

19. $f(z) = \tan z$

20. $f(z) = \dfrac{\cot \pi z}{z^2}$

21. $f(z) = \dfrac{1 - \cosh z}{z^4}$

22. $f(z) = \dfrac{e^z}{z^2}$

23. $f(z) = \dfrac{1}{1 + e^z}$

24. $f(z) = \dfrac{e^z - 1}{z^2}$

25. $f(z) = \dfrac{\sin z}{z^2 - z}$

26. $f(z) = \dfrac{\cos z - \cos 2z}{z^6}$

In Problems 27 and 28, show that the indicated number is an essential singularity of the given function.

27. $f(z) = z^3 \sin\left(\dfrac{1}{z}\right);\ z = 0$

28. $f(z) = (z - 1)\cos\left(\dfrac{1}{z + 2}\right);\ z = -2$

29. Determine whether $z = 0$ is an essential singularity of $f(z) = e^{z+1/z}$.

30. Determine whether $z = 0$ is an isolated or non-isolated singularity of $f(z) = \tan(1/z)$.

Focus on Concepts

31. In part (b) of Example 2 in Section 6.3, we showed that the Laurent series representation of $f(z) = \dfrac{1}{z(z-1)}$ valid for $|z| > 1$ is

$$f(z) = \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \frac{1}{z^5} + \cdots .$$

The point $z = 0$ is an isolated singularity of f, and the Laurent series contains an infinite number of terms involving negative integer powers of z. Discuss: Does this mean that $z = 0$ is an essential singularity of f? Defend your answer with sound mathematics.

32. Suppose f and g are analytic functions and f has a zero of order m and g has zero of order n at $z = z_0$. Discuss: What is the order of the zero of fg at z_0? of $f + g$ at z_0?